The Public Good Provision Problem Re-Examined

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(Latest Version Here)

Abstract

I write down a government's public good provision problem from first principles and, contrary to popular wisdom, I find a solution. I call it the cost-sharing pivotal mechanism. Both the statement of the problem and the solution are new. The cost-sharing pivotal mechanism satisfies a new participation constraint, a new fairness principle, and is strategy-proof, efficient, and asymptotically ex-post budget-balanced in large populations. Moreover, I show that a commonly used methodological simplification in mechanism design is not without loss, standard participation constraints used in mechanism design are not well-suited for these environments, and the most well-known mechanism for public good provision, the Clarke mechanism, violates a basic fairness constraint—if nothing is produced, no one should pay.

1 Introduction

A group of individuals would like to decide how much of a costly public good to produce and how it should be funded. Ideally, the procedure they employ would (1) provide incentives for each individual to truthfully reveal their value for the public good, (2) provide incentives for each individual to participate, (3) produce the socially optimal amount of the public good, (4) raise exactly enough revenue to produce this amount, and (5) not tax anyone unfairly. Call this the *public good provision problem*. It is well known that no decision procedure can satisfy all at once—indeed, Green and

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Laffont (1979) show that no mechanism can satisfy (1), (3), and (4) alone. I show that, in large populations, there are procedures that come arbitrarily close.

In particular, I propose a mechanism which satisfies (1), (2), (3), and (5) exactly and (4) arbitrarily closely as the population gets large. Formally, these properties correspond to (1) strategy-proofness, (2) cost-sharing universal participation, (3) efficiency, (4) asymptotic ex-post budget-balance, and (5) the fair pricing principle. (1), (3), and (4) are standard properties in the literature. (2) and (5) are new desiderata proposed in this paper. *Cost-sharing universal participation* states that each individual should always prefer to participate in the mechanism rather than to not participate—that is, rather than to receive the alternative chosen without her and to be taxed her fair share of its cost. The *fair pricing principle* states that each individual should never pay more than the maximum of her value for the chosen alternative and her fair share of its cost.[1]

This mechanism can be seen as a generalization of the celebrated pivotal mechanism[2] to environments with implementation costs—environments where each social alternative has some cost of implementation or production—and as such I call it the *cost-sharing pivotal mechanism*. The mechanism is simple, intuitive, and detail-free.[3]

> **The cost-sharing pivotal mechanism** employs an efficient decision rule and charges each individual her fair share of the cost of the decision that would have been made without her plus the welfare loss imposed on others by taking her preferences into account for the decision.

In other words, in a cost-sharing pivotal mechanism, the efficient level of the public good is produced and each individual pays her fair share of the cost of what is produced, unless she is pivotal,[4] in which case she pays her fair share of the cost of *what would have been produced without her* plus enough to compensate the others for the welfare loss imposed on them.

Importantly, this is not equivalent to the mechanism constructed via the standard approach—the net value approach—to modeling implementation costs in these set-

[1] An individual's fair share—what is, at the outset, considered a fair amount for an individual to contribute towards the public good—is agreed upon in advance and can depend on any observable traits including income level and distance from the public good. See Section 4 for a complete discussion.

[2] The pivotal mechanism (also called the VCG mechanism after Vickrey (1961), Clarke (1971), and Groves (1973)) is one mechanism within the Groves class—the class of strategy-proof and efficient mechanisms in quasilinear environments with private values. I define it informally in Section 2.3 and formally in Section 5.

[3] A mechanism is *detail-free* if it does not depend on the distribution of values in the population nor on any individual's beliefs.

[4] An individual is *pivotal* if the decision made with her differs from the decision made without her.

tings. This produces the Clarke mechanism.[5] I argue that the net value approach can be shortsighted and, in particular, that the Clarke mechanism violates an extremely weak and intuitive criterion—if nothing is produced, no one should pay.[6] I call this *no-extortion*.

The key takeaways from this paper, ordered sequentially, are fourfold.

1. The standard approach to modeling implementation costs in mechanism design—embedding them within the individuals' values (the *net value approach*)—is not without loss of generality, contrary to popular wisdom.[7] Keeping values and costs separate provides a richer mathematical and conceptual structure, allowing for a wider class of desiderata, mechanisms, and formal results—all of which this paper builds upon.

2. If a governing body has the power to tax, standard participation constraints (individual rationality and universal participation) are ill-suited desiderata for public good provision mechanisms. A new participation constraint (*cost-sharing universal participation*) and a new fairness principle (the *fair pricing principle*) should be used instead.

3. A new mechanism, which I call the *cost-sharing pivotal mechanism*, is a solution to the public goods problem for large populations and is effectively the unique solution. It satisfies strategy-proofness, efficiency, cost-sharing universal participation, the fair pricing principle, no-extortion, and asymptotic ex-post budget-balance.

4. The best-known proposed solution, the *Clarke mechanism*, violates cost-sharing universal participation, the fair pricing principle, and no-extortion.

The rest of the paper is organized as follows. Section 2 contains a summary of the paper and can be read as a complete description of the main ideas and results with only the necessary formalism to convey them precisely. In particular, Sections 2.1, 2.2, 2.3, and 2.4 present each of the above key takeaways in turn. Section 3 contains a review of the related literature. Sections 4, 5, 6, 7, and 8 contain the formal model and results. I introduce each concept and prove each result within the most general environment in which it applies. The sections are in decreasing order of generality.

In Section 4, I define mechanism design environments in which implementation costs are explicitly modeled.

[5] The Clarke mechanism was first proposed by Clarke (1971). The terms pivotal mechanism and Clarke mechanism are used interchangeably in the literature—see e.g., Mas-Colell, Whinston and Green (1995, p. 878). In this paper, I define them separately. See Sections 2.3 and 2.4 for an informal discussion and Sections 5 and 6 for a formal discussion.

[6] See Example 1.

[7] See Green and Laffont (1979, p. 31).

In Section 5, I consider environments with quasilinear preferences and private and convex values and I introduce a generalized participation constraint—cost-sharing universal participation—and use it to characterize the cost-sharing pivotal mechanism (Theorem 1).

In Section 6, I discuss the standard approach to modeling implementation costs in the literature—embedding cost shares into the individual's values and, in doing so, removing explicit implementation costs from the environment. I define the Clarke mechanism and discuss the properties it satisfies, the properties it violates, and why. I introduce a minimal fairness principle—no-extortion. I show that no strategy-proof and efficient mechanism can satisfy no-deficit and no-extortion (Theorem 2). The Clarke mechanism satisfies strategy-proofness, efficiency, and no-deficit and hence violates no-extortion.

In Section 7, I consider environments with a unidimensional space of social alternatives with continuous and non-decreasing values.[8] I introduce two new fairness principles—the fair pricing principle and the fair pricing principle per unit—and use them to characterize the cost-sharing pivotal mechanism (Theorem 3). I show that the cost-sharing pivotal mechanism is undominated, in that no suitable mechanism always comes closer to ex-post budget-balance (Theorem 4).

In Section 8, I consider environments with an ordered and finite space of social alternatives with non-decreasing values. I show that the cost-sharing pivotal mechanism is asymptotically ex-post budget-balanced (Theorem 5) and that it is effectively the unique such suitable mechanism (Theorem 6).

Section 9 concludes. All proofs are included in the main body or in Appendix B. A proof sketch or commentary is provided in the main body whenever the full proof is omitted.

2 Key Insights and Main Results

In this section, I discuss each of the four key takeaways of the paper. This encompasses all of the main results and can be taken as a complete summary of the paper with only the necessary formalism to convey the ideas precisely.

[8] This is the canonical and most general case of a single public good provision problem, whose provision levels can be infinitely fine-grained, and for whom all individuals cannot be made worse off by higher provision levels (the public good cannot be a bad).

2.1 The net value approach is not without loss of generality

The standard approach to modeling implementation costs in mechanism design is to exogenously assign cost shares to individuals, embed these cost shares into their value for the alternatives, and proceed with the analysis as if there were no implementation costs to begin with. Each individual is required to pay their share of the cost of whatever alternative is chosen, and they simply report their value for each alternative net of this amount. This quantity is called the individual's *net value*, and we may call this approach the *net value approach.* The idea is that it is without loss of generality to study environments without implementation costs, since implementation costs can always be baked into the alternatives in this way.[9] Indeed, Green and Laffont (1979, p. 31) even contend that "there is no real alternative to this approach." This paper shows otherwise. In particular, keeping values and costs separate provides a richer mathematical and conceptual structure which allows for a wider class of desiderata, mechanisms, and formal results than can be constructed under the net value approach. I discuss important examples of each below.

Universal participation is a desirable participation constraint in public good environments without implementation costs. Under the net value approach, there is just one way to generalize conditions from environments without implementation costs to those with them—by netting out costs from each individual's value and proceeding as if there were no implementation costs. Generalizing universal participation to such environments using the net value approach gives rise to a distinct condition I call *net-value universal participation.* As discussed in Sections 2.2 and 6, this condition is rather unnatural. By contrast, keeping values and costs separate allows for a wider space of generalizations, and I propose that the most appropriate generalization of universal participation to environments with implementation costs is instead a new participation constraint I call *cost-sharing universal participation.*

Similarly, the pivotal mechanism is a desirable mechanism in public good environments without implementation costs. Under the net value approach, there is just one way to generalize mechanisms from environments without implementation costs to those with them. Generalizing the pivotal mechanism to such environments using the net value approach gives rise to the *Clarke mechanism.* As discussed in Sections 2.4 and 6, the Clarke mechanism violates several reasonable participation and fairness criteria. By contrast, keeping values and costs separate allows first, for a wider space of generalizations and second, for a more careful and precise specification of the public good provision problem itself. These insights lead to a relatively natural solution—a new mechanism I call the *cost-sharing pivotal mechanism.* Defining the public good provision problem in this way and showing how and why the cost-sharing pivotal

[9] See Green and Laffont (1979, p. 29). See also Moulin (1988, p. 205), Varian (1992, p. 426), and Mas-Colell, Whinston and Green (1995, p. 877).

mechanism is a natural solution, and effectively the unique solution, is a primary contribution of this paper.

Finally, the net value approach significantly constrains the generality of large population results. In these results, one supposes that values are i.i.d. draws from some distribution and derives properties of the mechanism in question for environments with large populations. If we instead suppose that *net values* are i.i.d. draws from some distribution, this bakes in the additional assumption that 1) cost shares are split equally across individuals and 2) costs increase linearly with population size. As discussed in Section 8, these are significant assumptions and should be avoided whenever possible. On the other hand, by keeping values and costs separate I avoid these assumptions and, moreover, show that the large population results for the cost-sharing pivotal mechanism are robust to *any* specification of cost shares—including those which are constructed from observable data such as income, which may be correlated with individual values—and *any* relation between costs and population size.

2.2 A new participation constraint and fairness principle for public good provision

Participation constraints can be interpreted both as giving incentives for participation and as capturing some notion of fairness. Under either interpretation, standard participation constraints are not well-suited for public good provision environments in which a government has the power to tax. I propose a new participation constraint called *cost-sharing universal participation* and a new fairness principle called the *fair pricing principle.*

A participation constraint requires that each individual prefers to participate in the mechanism rather than to not participate and receive some outside option. The most widely-used participation constraint is individual rationality, where the outside option is some fixed outcome with a payoff of zero.

> **Individual Rationality.** Each individual should always prefer to participate in the mechanism rather than to not participate—that is, rather than to consume nothing and to pay nothing.

This is fitting, for example, in an auction setting, where the outcome from not participating is in fact to consume and pay nothing. However, it is not fitting in a public goods setting, where each individual consumes what the mechanism produces no matter if they participate or not—since public goods are by definition non-rival and non-excludable. If there is no governing body with the power to tax (or there are no implementation costs), this fact is captured by a participation constraint known as universal participation.[10]

[10] See Green and Laffont (1979, Chapter 6) for a similar discussion of individual rationality and universal participation. Universal participation is sometimes called *no-free-ride*—see e.g., Moulin

Universal Participation. Each individual should always prefer to participate in the mechanism rather than to not participate—that is, rather than to receive the alternative chosen without her and to pay nothing.

That said, if there is a governing body with the power to tax (and there are implementation costs), not participating need not result in zero taxes. In practice, many public goods are funded by tax dollars which violate universal participation (e.g., my tax dollars may go to fund a park for which I have zero value). I propose the following condition which captures a natural participation criterion for public good environments when a government has the power to tax.

Cost-Sharing Universal Participation. Each individual should always prefer to participate in the mechanism rather than to not participate—that is, rather than to receive the alternative chosen without her and to be taxed her fair share of its cost.

Each individual's fair share of the cost of a public good is agreed upon in advance and can in principle depend on any observable traits (including income, physical distance from the good, etc.)—just not the individual's report of her value for the good itself.[11] Cost-sharing universal participation simply appends to universal participation that, in addition to receiving what is produced without her, an individual is also taxed her fair share of its cost.

Instead of explicitly defining a constraint that contains values and costs separately, it is common in the literature to use the net value approach and to embed costs into each individual's value for the alternatives. Because universal participation is a desirable constraint without implementation costs, it is often simply presumed that a mechanism which satisfies universal participation will continue to be desirable when adding implementation costs in this way. In fact, to my knowledge nobody has taken the step of explicitly spelling out what universal participation states when net values are employed. Since doing so indeed produces a distinct constraint, I give it its own name—*net-value universal participation*. It states the following.

Net-Value Universal Participation. Each individual should always prefer to participate in the mechanism rather than to not participate—that is, rather than to receive the alternative chosen had she valued each good at precisely her fair share of its cost and to be taxed her fair share of the cost of that good.

The difference between cost-sharing universal participation and net-value universal participation is what the mechanism purports to do when an individual does not participate. In the former, the mechanism selects the optimal decision supposing the individual had a value of zero for each good and charges them their fair share of

(1986).

[11] See Section 4 for a complete discussion.

its cost, while in the latter, the mechanism selects the optimal decision supposing the individual had a value exactly equal to their fair share of the cost of each good, and charges them their fair share of the cost of that good. In my view, the latter is an unnatural desiderata. Both are generalizations of universal participation to environments with implementation costs.[12] The former retains the idea from universal participation that the decision made if an individual does not participate is equivalent to the decision made had they never existed. The latter does not.

Participation constraints can also be interpreted as principles of fairness. Individual rationality is a particularly strong one. Reinterpreted as a fairness principle, it says the following.

> **Individual Rationality.** Each individual should never pay more than her value for the chosen alternative.

Individual rationality states that it is fair for an individual to pay up to her total value of what is produced and no more. This is arguably too demanding for public good environments. Such environments are commonly characterized by a sense of community, wherein each individual understands that everyone needs to chip in—but not unreasonably so—for the greater good, even if that means paying more than their value. I propose the following principle, which weakens individual rationality, to formalize this view.

> **Fair Pricing Principle.** Each individual should never pay more than the maximum of her value for the chosen alternative and her fair share of its cost.

The fair pricing principle states that 1) it is fair for an individual to pay up to her total value of what is produced, and 2) it is fair for an individual to pay up to her fair share of the cost of what is produced. This captures the sentiment that it is fair for an individual to pay more than her fair share—as long as she is made better off by the mechanism—and for an individual to be made worse off by the mechanism—as long as she pays less than her fair share.

Both cost-sharing universal participation and the fair pricing principle are desirable criteria for public good provision mechanisms. As we will see in the next section, imposing one, the other, or both yields the same solution, which I call the *cost-sharing pivotal mechanism*.

2.3 The cost-sharing pivotal mechanism

The cost-sharing pivotal mechanism is a generalization of the pivotal mechanism to environments with implementation costs.

[12] With no implementation costs, all three are equivalent.

> **The pivotal mechanism** employs an efficient decision rule and charges each individual the welfare loss imposed on others by taking her preferences into account for the decision.

In a pivotal mechanism, there is no payment related to the implementation costs of the alternatives. Indeed, if no one is pivotal, no one pays. The cost-sharing pivotal mechanism simply adds that, in addition to her pivotal payment, each individual also pays her fair share of the cost of the alternative that would have been chosen without her. If no one is pivotal, everyone pays their fair share of the chosen alternative.

> **The cost-sharing pivotal mechanism** employs an efficient decision rule and charges each individual her fair share of the cost of the decision that would have been made without her plus the welfare loss imposed on others by taking her preferences into account for the decision.

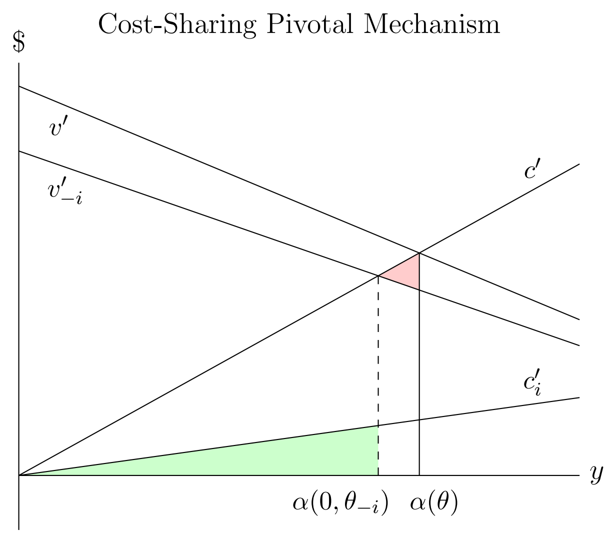

Figure 1

In Figure 1, y is the quantity of the public good, v' is the social marginal value of the public good, v'_{-i} is the social marginal value minus i's, c' is the marginal cost of the public good, c'_i is i's marginal fair share of the cost, $\alpha(\theta)$ is the efficient decision (where v' and c' cross), and $\alpha(0, \theta_{-i})$ is the efficient decision ignoring i's preferences (where v'_{-i} and c' cross). The cost-sharing pivotal mechanism implements $\alpha(\theta)$ and charges i her fair share of the cost of the decision that would have been made without her (area in green) plus the welfare loss imposed on others by taking her preferences into account for the decision (area in red). The pivotal mechanism charges i only the latter (area in red).

The pivotal mechanism is one particular mechanism in the class of Groves mechanisms—the class of all strategy-proof and efficient mechanisms. It is well known that no mechanism is strategy-proof, efficient, and ex-post budget-balanced.[13] However, some mechanisms come closer to budget-balance than others. The cost-sharing pivotal mechanism is another mechanism in the class of Groves mechanisms, but unlike the pivotal mechanism, it comes arbitrarily close to ex-post budget-balance as the population gets large.

> **Asymptotically Ex-Post Budget-Balanced.** The probability of ex-post budget-balance goes to one and the expected distance from ex-post budget-balance per capita goes to zero as the population size goes to infinity.

It is common in the literature on public good provision to restrict attention to mechanisms which never run a deficit. This is often simply termed *feasibility.* I argue that in public good provision environments with large populations, this is an inappropriate notion of feasibility. In such environments, surpluses and deficits must ultimately trickle down to the citizens. What matters is that their incentives are preserved in light of this fact. This inspires the forthcoming principle.

I propose that governments can finance small deficits and distribute small surpluses through future changes to tax policy which are sufficiently inconsequential so as not to affect the incentives of their citizens. That is, violations of budget-balance in either direction are feasible, as long as they are "small". One way to understand this is that governments maintain a fund capable of absorbing small deficits and surpluses. Over time, any accumulated surplus is repaid through tax cuts, while any accumulated deficit is replenished through tax increases. If these adjustments are small, it is reasonable to assume that individuals do not perceive them as part of the mechanism. I call this the *fuzzy taxation principle.*[14]

> **Fuzzy Taxation Principle.** Sufficiently small, indirect, and distant[15] changes to tax policy are not perceived by individuals as part of their transfer.

Under the fuzzy taxation principle, if the expected distance from ex-post budget-

[13] A mechanism is *strategy-proof* if it is a dominant strategy for each individual to report their value truthfully, *efficient* if the decision maximizes total value minus implementation costs, and *ex-post budget-balanced* if the revenue generated exactly offsets the implementation cost of the decision made. The Groves mechanisms were first proposed by Groves (1973). Holmström (1979) showed that a mechanism is strategy-proof and efficient if and only if it is a Groves mechanism (on a suitably rich domain). Green and Laffont (1979) show that no mechanism is strategy-proof, efficient, and ex-post budget-balanced (on a suitably rich domain).

[14] This is not to be confused with the *taxation principle*. The taxation principle states that a strategy-proof mechanism must assign the same transfer to any two reports that result in the same decision (Salanié, 2017, p. 18). The fuzzy taxation principle states that small changes to future tax policy do not affect individual incentives.

[15] Changes are both distant in context (changes to tax policies not related to public good provision mechanisms) and in time (changes occur sufficiently far into the future).

balance per capita is sufficiently small, the mechanism is feasible. Note that this is a not a formal criterion itself, but rather a rationale for using suitable formal criteria. In particular, the fuzzy taxation principle is a rationale for the desirability of asymptotic ex-post budget-balance and the irrelevance for feasibility of a public good provision mechanism never running a budget deficit.

The cost-sharing pivotal mechanism satisfies strategy-proofness, efficiency, cost-sharing universal participation, the fair pricing principle, and asymptotic ex-post budget-balance (Theorems 1, 3, and 5), and it is effectively the unique mechanism which does so (Theorems 4 and 6). In particular, Theorem 1 shows that a mechanism maximizes ex-post revenue among all mechanisms which satisfy strategy-proofness, efficiency, and cost-sharing universal participation if and only if it is a cost-sharing pivotal mechanism. Theorem 3 shows that a mechanism maximizes ex-post revenue among all mechanisms which satisfy strategy-proofness, efficiency, and the fair pricing principle if and only if it is a cost-sharing pivotal mechanism. Theorem 4 shows that there is no mechanism that satisfies strategy-proofness, efficiency, and at least one of cost-sharing universal participation and the fair pricing principle which always gets closer to ex-post budget-balance than a cost-sharing pivotal mechanism. Theorem 5 shows that the cost-sharing pivotal mechanism is asymptotically ex-post budget-balanced. Theorem 6 shows that any mechanism which satisfies strategy-proofness, efficiency, asymptotic ex-post budget-balance, and at least one of cost-sharing universal participation and the fair pricing principle, converges to a cost-sharing pivotal mechanism as the population gets large.

2.4 The Clarke mechanism violates no-extortion

The Clarke mechanism is an alternative way to generalize the pivotal mechanism to environments with implementation costs. It does so using the net value approach—by embedding an individual's fair share of the cost into her value for the alternatives and applying the pivotal mechanism to the resulting environment with no implementation costs. In the same way we generalized universal participation to net-value universal participation, we can generalize the pivotal mechanism to the "net-value pivotal mechanism". This is precisely the Clarke mechanism.

> **The Clarke mechanism** employs an efficient decision rule and charges each individual her fair share of the cost of what is actually chosen plus the modified welfare loss—her share of the cost removed—imposed on others by taking both her preferences and her share of the cost into account for the decision.

This definition is clunky, and unavoidably so. The pivotal mechanism is simple and elegant, but plugging net values into the pivotal mechanism and being explicit about the result is much less so. As with net-value universal participation, to my knowledge this is the first time the Clarke mechanism has been presented in this way.

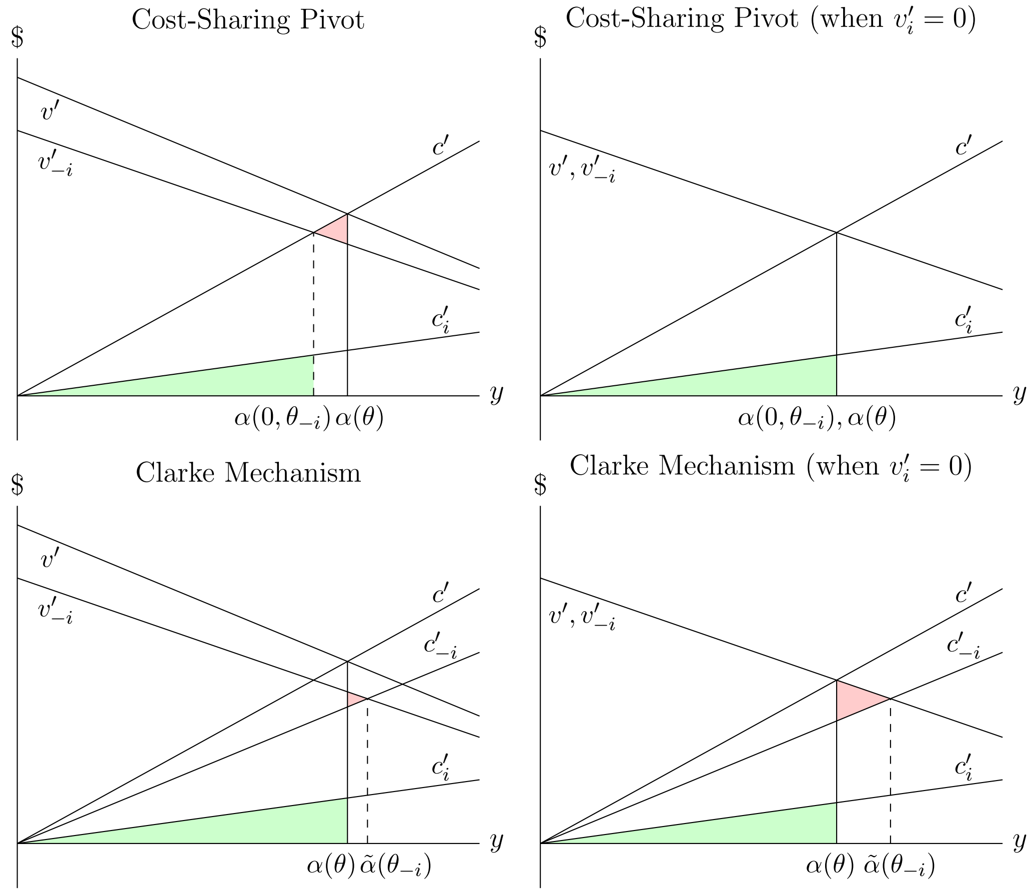

Figure 2

How do the cost-sharing pivotal mechanism and the Clarke mechanism compare? Figure 2 is similar to Figure 1, but adds c'_{-i}, the marginal cost of the public good minus i's share of the cost, and $\tilde{\alpha}(\theta_{-i})$, the efficient decision ignoring i's preferences and i's share of the cost (where v'_{-i} and c'_{-i} cross).[16] The Clarke mechanism charges i her fair share of the cost of what is actually chosen (area in green) plus the modified welfare loss—her share of the cost removed—imposed on others by taking both her preferences and her share of the cost into account for the decision (area in red).

As the right panel of Figure 2 shows, the Clarke mechanism incorporates a somewhat unusual notion of what it means for an individual to be pivotal. Suppose i has zero value for the public good. She is *not* pivotal—the efficient decision is the same with or without her. The cost-sharing pivotal mechanism charges i her fair share of the cost of what is produced (the green triangle). The Clarke mechanism charges i this amount plus a modified conception of a "pivotal" payment (the red triangle). Ignoring

[16] Formally, $\tilde{\alpha}(\theta_{-i}) \in \arg\max_{y \in Y} [\sum_{j \neq i} v_j(y, \theta_j) - c_j(y)]$ and $\alpha(\theta) \in \arg\max_{y \in Y} \sum_{i \in I} [v_i(y, \theta_i)] - c(y)$. Note that while $\tilde{\alpha}(\theta_{-i}) > \alpha(\theta)$ in both cases above, it may also be that $\tilde{\alpha}(\theta_{-i}) \leq \alpha(\theta)$.

i's preferences does not change the optimal decision ($\alpha(0, \theta_{-i}) = \alpha(\theta)$), but ignoring i's share of the cost makes the good cheaper, and if the good is cheaper, the others want to produce more of it ($\tilde{\alpha}(\theta_{-i}) > \alpha(\theta)$). The red triangle captures this "loss" to others—the welfare loss of not getting to produce more of the good if it were cheaper by precisely i's cost share. In the Clarke mechanism, an individual is "pivotal" if the efficient decision changes when excluding her preferences *and* her share of the cost.

This distinctive feature of the Clarke mechanism ultimately leads it to violate desirable participation constraints and fairness principles.[17] In particular, the Clarke mechanism violates cost-sharing universal participation, the fair pricing principle, and, notably, an especially weak criterion I call *no-extortion.* No-extortion applies the fair pricing principle only to the case when none of the public good is produced. It requires that when nothing is produced, no payments are collected. By contrast, the cost-sharing pivotal mechanism satisfies all three.

No-Extortion. If nothing is produced, no payments are collected.

Example 1 (Clarke mechanism violates no-extortion)**.** Let $I = \{i, j\}$ and $Y = \{0, 1\}$, where 1 represents the construction of a park and 0 represents no construction. The cost of the park is 4, and each individual's fair share of the cost is 2. Individual i values the park at 0, and individual j values the park at 3. The efficient decision is not to construct the park (in which case i's opponents' total modified welfare—i's share of the cost removed—is 0) and the efficient decision ignoring i's preferences *and* i's share of the cost is to construct the park (in which case i's opponents' total modified welfare—i's share of the cost removed—is 1), so i's Clarke transfer is 1.

This is equivalent to embedding cost shares into individual values, removing all implementation costs, and running a pivotal mechanism. Individual i values the park net of her cost share at -2, and individual j values the park net of her cost share at 1. Taking these as their values, the efficient decision is not to construct the park (in which case i's opponents' total welfare is 0) and the efficient decision ignoring i's preferences is to construct the park (in which case i's opponents' total welfare is 1), so i's pivotal transfer is 1. $\square$

One of the main selling points of the Clarke mechanism is that it never runs a budget deficit. Nevertheless, the fuzzy taxation principle argues that what really matters for the public good provision problem is that future budget-balancing policies do not distort individual incentives, and what really matters for *this* is that the expected distance from budget-balance is small. It is not particularly important that a mechanism always runs a budget surplus. Arguably, this is even a red herring. In fact,

[17] The pivotal mechanism satisfies universal participation, and so by construction, the Clarke mechanism satisfies net-value universal participation—but, as discussed in Section 2.2, this condition is arguably not relevant or appealing.

it turns out that never running a budget deficit and fairness are fundamentally in conflict with each other. No strategy-proof and efficient mechanism can satisfy both no-deficit and no-extortion (Theorem 2).

3 Related Literature

In this section, I review some of the related literature on the public good provision problem. Mailath and Postlewaite (1990) consider binary public good environments and show that in any mechanism satisfying Bayesian incentive-compatibility, interim individual-rationality, and ex-ante no-deficit, if the cost of the public good increases linearly with population size or faster, the probability that the public good is produced goes to zero as the population size goes to infinity—even when the probability that it is efficient to produce the public good converges to one.[18] In a similar vein, Al-Najjar and Smorodinsky (2000) consider binary public good environments and show that any mechanism satisfying Bayesian incentive-compatibility, interim individual-rationality, and "no small contributors"[19] cannot raise revenues that are unbounded as the population size goes to infinity.

Xi and Xie (2021) consider binary public good environments and propose a class of mechanisms which are strategyproof and ex-post individually rational. They show that if the cost of provision grows slower than the square root of population size, these mechanisms generate an ex-ante budget-surplus asymptotically and are asymptotically efficient, while if the cost grows faster than the square root of population size, any mechanism which is strategypoof, ex-post individually-rational, and generates an ex-ante budget-surplus asymptotically must have a provision probability converging to zero. Kuzmics and Steg (2017) consider binary public good environments and show that the welfare-maximizing mechanism among all strategy-proof, ex-post individually rational, and no-deficit mechanisms is a "split-the-cost" mechanism, in which each player has a fixed cost share, the good is provided if and only if all players' values exceed their own cost share, and each player pays her cost share if the good is provided and zero otherwise. Notice that this is a special case of Mailath and Postlewaite (1990), and hence if the cost of the public good increases linearly with population size or faster, the probability that the public good is produced in such a mechanism goes to zero as the population size goes to infinity.

Serizawa (1999) considers continuous public good environments with monotonic and quasi-concave preferences and characterizes the class of strategy-proof and ex-post budget-balanced mechanisms which are also symmetric (minimax rule), anonymous (q-rule), or symmetric and ex-post individually-rational (minimum demand rule).

[18] See also Hellwig (2003).

[19] This condition requires that, if an individual's expected transfer is less than some threshold, then the individual's expected transfer is zero.

Laffont and Maskin (1982) show that in binary decision problems with no implementation costs, if the population distribution of values is symmetric around zero, the welfare-maximizing budget-balanced mechanism has a sink agent—an agent who's preferences are ignored and who receives the transfers from the other agents—and a decision rule which is efficient among the remaining agents. Nath and Sandholm (2019) consider finite decision problems with three or more alternatives and no implementation costs and bound the asymptotic inefficiency of strategy-proof mechanisms using strategy-proof and ex-post budget-balanced mechanisms with a sink agent and a decision rule which is efficient among the non-sink agents. Moreover, they show that any strategy-proof and ex-post budget-balanced mechanism has at least one a sink agent. Drexl and Kleiner (2018) consider binary decision problems with no implementation costs and show that the welfare-maximizing mechanism among all anonymous, strategy-proof, universal participation, and no-deficit mechanisms is qualified majority voting with threshold k, where k is the ceiling of $\frac{-n\mathbb{E}(\theta_i \mid \theta_i \leq 0)}{\mathbb{E}(\theta_i \mid \theta_i \geq 0) - \mathbb{E}(\theta_i \mid \theta_i \leq 0)}$ and θ_i is i's value for the project. Note that the use of anonymous mechanisms rules out having a sink agent.

Rob (1982) considers binary decision environments with no implementation costs and shows that the pivotal mechanism is asymptotically efficient—i.e., the probability that any individual has a positive transfer goes to zero and the expected per capita transfer goes to zero as the population size goes to infinity.[20] As discussed in Sections 2.1 and 8, this immediately implies an analogous result for the Clarke mechanism in a binary public good environment with implementation costs (using the net value approach to embed costs into values) with equal cost shares and a cost of the public good which increases linearly with population size. By contrast, I show that the cost-sharing pivotal mechanism is asymptotically ex-post budget-balanced in a finite public good environment with arbitrary sequences of costs and arbitrary sequences of cost shares. See Sections 4 and 8 for a discussion of the importance of this additional robustness.

4 Model

Implementation costs are often not explicitly modeled in mechanism design.[21] This paper is built on insights obtained by doing just that. A minimal *mechanism design environment with implementation costs* is given by

$$\mathcal{E} = (I, X, \Theta, \{u_i\}_{i \in I}, c),$$

where I is a set of n individuals, $X = Y \times \mathbb{R}^n$ is a set of outcomes consisting of a social alternative $y \in Y$ and an n-vector of monetary transfers (where $t_i \in \mathbb{R}$ is the

[20] See also Green, Kohlberg and Laffont (1976), Green and Laffont (1979), and Mitsui (1983).

[21] See Sections 2.1 and 6 for a discussion.

amount i is asked to pay), $\Theta = \Theta_i \times \ldots \times \Theta_n$ is a type space, $u_i : X \times \Theta \to \mathbb{R}$ specifies i's payoff for each outcome given a type profile θ, and $c : Y \to \mathbb{R}$ specifies the implementation cost of each alternative.

In this paper, we will consider a *mechanism design environment with implementation costs and fair cost shares*, given by

$$\mathcal{E} = (I, X, \Theta \times Z, \{u_i\}_{i \in I}, c, \phi, F).$$

As before, $\Theta = \Theta_1 \times \ldots \times \Theta_n$ is a type space and $u_i : X \times \Theta \to \mathbb{R}$ specifies i's payoff for each outcome given a type profile θ. Meanwhile, $Z = Z_1 \times \ldots \times Z_n$ is a space of auxiliary traits which are not directly payoff relevant—i.e., not payoff relevant conditional on θ. $\phi : Y \times Z \to \mathbb{R}^n$ is a mapping from Z into what society has deemed a fair share of the implementation cost of each alternative y for each individual i. Hence, $\sum_{i \in I} \phi_i(y, z) = c(y)$ for all y and z. It is often convenient to suppress the dependence on z and denote $c_i(y) \equiv \phi_i(y, z)$. $F \in \Delta(\Theta \times Z)$ is a prior distribution over $\Theta \times Z$. Let $\mathbb{E}$ be the set of all such environments.[22]

The notion of exogenous cost shares is standard in the literature.[23] That said, explicitly modeling them as an arbitrary function of auxiliary traits is new. While this is unnecessary for the non-asymptotic results in Sections 5, 6, and 7 (along with the prior over $\Theta \times Z$), doing so is integral for the asymptotic results in Section 8. In particular, assuming equal cost shares (which is necessary when using standard methods) assumes away several challenges which are true in practice—namely, that we may want richer individuals to have higher cost shares and that richer individuals may have a higher willingness to pay on average.

I now discuss the interpretation of these fair shares, including 1) why ϕ does not depend on θ and 2) if Z ought to include traits which are private information. Such a discussion is, to the best of my knowledge, new.

In my view, fairness in public good provision environments ought to capture a sense of community, wherein each individual has, and understands they have, a responsibility to play their part—regardless of their value for the good itself. The most elementary way to capture this responsibility is for a society to agree in advance that it is fair to ask any individual to contribute their *equal share* of the cost of a public good—i.e., the cost of the public good divided by the population size. Intuitively, if a public park is built in my city, it is fair for the government to ask me to pay an equal share of its cost, regardless of how much I value the park.

Extending this logic, a more sophisticated way to capture this responsibility is for a society to agree in advance that it is fair to ask any individual to contribute an amount

[22] I will also use $\mathbb{E}$ to denote the expectation operator. The intended meaning should be clear from context.

[23] See Green and Laffont (1979, p. 29), Moulin (1988, p. 205), Varian (1992, p. 426), and Mas-Colell, Whinston and Green (1995, p. 877).

which may depend on deliberately selected traits of that and other individuals which are deemed relevant to fairness. Let's call this amount an individual's *fair share*. Intuitively, if a public park is built, it is fair for the government to ask me to pay an amount which may be higher the closer I live to the park and the richer I am, regardless of how much I value the park. Importantly, these traits should not include my value for the public good. The sense of communal responsibility surrounding the provision of public goods is precisely that I may be asked to pay an amount that does not depend on my (or others') value for the good itself.[24]

Z is precisely this set of auxiliary traits. The question then remains: can and should Z include traits which are private information to the individuals? The answer is no. The taxation principle states that a strategy-proof mechanism must assign the same transfer to any two reports that result in the same decision (Salanié, 2017, p. 18). Since we seek an efficient decision rule and strategy-proof incentives, and z_i does not affect i's payoff conditional on θ_i, this implies that we cannot tailor i's transfer to i's reported z_i and hence cannot elicit z_i.

This point effectively ends the discussion, but it is nonetheless worth pointing out a second consideration. The selection of traits which determine each individual's fair share is a sensitive matter and ought to be done thoughtfully by an interdisciplinary team of researchers, policy makers, and ordinary citizens. A good heuristic here seems to be that selected traits should be directly observable by the government (rather than reported through an incentive mechanism). The two traits used as primary examples in this paper, income and distance from the public good, fit this provision. See Appendix A for explicit examples of how a government might construct fair shares from observable traits—i.e., explicit examples of Z and ϕ.

Hence, a (direct revelation) mechanism $f : \Theta \to X$ is a mapping from type profiles to outcomes. Restricting attention to direct revelation mechanisms is without loss of generality by the revelation principle for dominant strategies (Gibbard, 1973). In these environments, f can be represented by a pair $f = (\alpha, \tau)$, where $\alpha : \Theta \to Y$ is a decision rule and $\tau : \Theta \to \mathbb{R}^n$ is a transfer rule. Each individual i reports their type θ_i to the mechanism. The mechanism then implements social alternative $\alpha(\theta)$ at cost $c(\alpha(\theta))$ and collects transfers $\tau_i(\theta)$ from each i.

5 Convex Environments

Let $\mathbb{E}_X \subset \mathbb{E}$ be the set of environments with quasilinear preferences, private and convex values, and a fully indifferent type.[25] In particular, $\mathbb{E}_X$ is the set of environments

[24] The fair pricing principle, discussed in Sections 2.2 and 7, uses these fair shares to construct a fairness criterion for public good provision. It states that it is fair for the government to ask an individual to pay up to the maximum of their value for the good and their fair share of its cost.

[25] X stands for "convex".

where each of the following are true. For each i and θ, $u_i(x, \theta) = v_i(y, \theta_i) - t_i$ for some $v_i : Y \times \Theta_i \to \mathbb{R}$. For all i, $\lambda \in [0, 1]$, and $\theta'_i, \theta''_i \in \Theta_i$, there exists $\theta_i \in \Theta_i$ such that for all y, $v_i(y, \theta_i) = \lambda v_i(y, \theta'_i) + (1 - \lambda) v_i(y, \theta''_i)$. For all i, there exists a type, denoted $0 \in \Theta_i$, such that $v_i(y, 0) = 0$ for all y. For every θ, there exists an efficient social alternative, i.e., $\arg\max_{y \in Y} \sum_{i \in I} v_i(y, \theta_i) - c(y)$ exists.

A mechanism is strategy-proof if it is a dominant strategy for each individual i to report her type θ_i truthfully to the mechanism.[26]

Definition 1.1. A mechanism $f = (\alpha, \tau)$ is *strategy-proof* if for all i, θ'_i, θ,

$$v_i(\alpha(\theta), \theta_i) - \tau_i(\theta) \geq v_i(\alpha(\theta'_i, \theta_{-i}), \theta_i) - \tau_i(\theta'_i, \theta_{-i}).$$

Let $A^*(\theta) = \arg\max_{\ y \in Y} \sum_{i \in I} v_i(y, \theta_i) - c(y)$ be the set of efficient decisions[27] given θ.

Definition 1.2. A decision rule $\alpha : \Theta \to Y$ is *efficient* if for all θ, $\alpha(\theta) \in A^*(\theta)$. A mechanism $f = (\alpha, \tau)$ is *efficient* if α is efficient.

For any convex environment, the class of Groves mechanisms fully characterizes the set of strategy-proof and efficient mechanisms (Holmström, 1979).

Definition 1.3. A mechanism $f = (\alpha, \tau)$ is a *Groves mechanism* if the decision rule is efficient and the transfer rule satisfies

$$\tau_i(\theta) = g_i(\theta_{-i}) - \Big[\Big(\sum_{j \neq i} v_j(\alpha(\theta), \theta_j)\Big) - c(\alpha(\theta))\Big]$$

for any function $g_i : \Theta_{-i} \to \mathbb{R}$.

The term in square brackets is the welfare attained by i's opponents, and g_i is any function that does not depend on i's report (and hence does not affect her incentives). By subsidizing i an amount exactly equal to the welfare of her opponents, the planners' objective and i's objective become one and the same—to maximize social welfare.

Theorem (Holmström, 1979). *Given any environment $\mathcal{E}_X \in \mathbb{E}_X$, a mechanism is strategy-proof and efficient if and only if it is a Groves mechanism.*

[26] If a mechanism fails to be strategy-proof (or another suitable form of incentive-compatibility), we cannot count on the individuals to report truthfully and all the other properties we discuss, which rely on this, will be immaterial.

[27] Given quasilinear preferences on $X = Y \times \mathbb{R}^n$, for any θ, the set of efficient outcomes consists of any social alternative $y \in A^*(\theta)$ and any budget-balanced transfers $t \in \mathbb{R}^n$ where $\sum_i t_i = c(y)$. Regardless of whether a mechanism achieves budget-balance, it is common to refer to these alternatives as the *efficient decisions* and mechanisms which select these alternatives as *efficient mechanisms*.

One of the most celebrated mechanisms within the Groves class is the *pivotal mechanism.* Before defining it formally, I clarify an important point. The pivotal mechanism is generally defined in environments without implementation costs. In such environments, it is defined by $\alpha(\theta) \in \arg\max_{y \in Y} \sum_{i \in I} v_i(y, \theta_i)$ and $\tau_i(\theta) = \sum_{j \neq i} v_j(\alpha(0, \theta_{-i}), \theta_j) - \sum_{j \neq i} v_j(\alpha(\theta), \theta_j)$ for all i. This fits the natural language definition from Section 2.3: the pivotal mechanism employs an efficient decision rule and charges each individual the welfare loss imposed on others by taking her preferences into account for the decision.

It is standard to additionally call the mechanism generated by plugging net values into a pivotal mechanism (the net value approach to modeling implementation costs) a pivotal mechanism.[28] However, this no longer fits the natural language definition above. Instead, such a mechanism: employs an efficient decision rule and charges each individual her fair share of the cost of what is actually chosen plus the modified welfare loss—her share of the cost removed—imposed on others by taking both her preferences and her share of the cost into account for the decision. This is precisely the mechanism proposed by Clarke (1971), and as such I refer to it as the *Clarke mechanism*, which I define formally in Section 6.

What, then, is a pivotal mechanism in environments with implementation costs? Or more generally, how does any definition in environments without implementation costs apply to environments with them? In my view, the most natural way to incorporate implementation costs into an environment without them is to introduce an additional player 0 who's commonly known[29] preference reflects the implementation cost of each alternative—i.e., $v_0(y, \theta_0) = -c(y)$ for all y. In the case of the pivotal mechanism, including implementation costs then simply updates the efficient decision rule to be $\alpha(\theta) \in \arg\max_{y \in Y} \sum_{i \in I} v_i(y, \theta_i) - c(y)$, leaving everything else the same. This preserves the original natural language definition of the mechanism and gives rise to the following formal definition.

Definition 1.4. A mechanism $f = (\alpha, \tau)$ is a *pivotal mechanism* if the decision rule is efficient and the transfer rule satisfies

$$\begin{aligned}
\tau_i(\theta) &= \left[\max_{y \in Y} \Big(\sum_{j \neq i} v_j(y, \theta_j)\Big) - c(y)\right] - \left[\Big(\sum_{j \neq i} v_j(\alpha(\theta), \theta_j)\Big) - c(\alpha(\theta))\right] \\
&= \left[\Big(\sum_{j \neq i} v_j(\alpha(0, \theta_{-i}), \theta_j)\Big) - c(\alpha(0, \theta_{-i}))\right] - \left[\Big(\sum_{j \neq i} v_j(\alpha(\theta), \theta_j)\Big) - c(\alpha(\theta))\right].
\end{aligned}$$

An individual is pivotal if the efficient decision changes when taking her preferences into account.

[28] See e.g., Mas-Colell, Whinston and Green (1995, p. 878). Note also that the terms pivotal mechanism and Clarke mechanism are used interchangeably in the literature.

[29] Because this player has no private information, there is nothing to incentivize them to report and their transfer can just be zero.

Definition 1.5. Given a mechanism $f = (\alpha, \tau)$ and type profile $\theta \in \Theta$, an individual i is *pivotal* if $\alpha(0, \theta_{-i}) \neq \alpha(\theta)$.

The pivotal mechanism earns its name from the fact that it only charges individuals who are pivotal. It charges them precisely the welfare loss they impose on others. In particular, it charges i the difference between the maximum welfare achieved by the rest of the population when not taking her preferences into account and the maximum welfare achieved by the rest of the population when taking her preferences into account.

Universal participation captures the idea that a mechanism should incentivize each individual to participate in the mechanism rather than to not participate, receive the alternative chosen without her, and pay nothing. This is an appropriate participation constraint in public good environments in which there is no governing body with the power to tax (or no implementation costs).

Definition 1.6. A mechanism $f = (\alpha, \tau)$ satisfies *universal participation* if for all θ and i,

$$v_i(\alpha(\theta), \theta_i) - \tau_i(\theta) \geq v_i(\alpha(0, \theta_{-i}), \theta_i).$$

Fryxell (2023) shows that the pivotal mechanism can be characterized as the revenue-maximizing Groves mechanism subject to universal participation.

Theorem (Fryxell, 2023)**.** *Given any environment $\mathcal{E}_X \in \mathbb{E}_X$, a mechanism maximizes ex-post revenue among all mechanisms which satisfy strategy-proofness, efficiency, and universal participation if and only if it is a pivotal mechanism.*

I propose a new participation constraint, which I call cost-sharing universal participation, capturing the idea that a mechanism should incentivize each individual to participate in the mechanism rather than to not participate, receive the alternative chosen without her, and be taxed her fair share of its cost. This is an appropriate participation constraint in public good environments in which there is a governing body with the power to tax.

Definition 1.7. A mechanism $f = (\alpha, \tau)$ satisfies *cost-sharing universal participation (CS-UP)* if for all θ and i,

$$v_i(\alpha(\theta), \theta_i) - \tau_i(\theta) \geq v_i(\alpha(0, \theta_{-i}), \theta_i) - c_i(\alpha(0, \theta_{-i})).$$

The main contribution of this paper is to propose the following mechanism as the natural generalization of the pivotal mechanism to environments with implementation costs and as a solution to the public good provision problem for large populations as defined in Section 1.

Definition 1.8. A mechanism $f = (\alpha, \tau)$ is a *cost-sharing pivotal mechanism (CSP)* if the decision rule is efficient and the transfer rule satisfies

$$\begin{aligned}\tau_i(\theta) &= \Big[\max_{y \in Y}\Big(\sum_{j \neq i} v_j(y, \theta_j)\Big) - c(y)\Big] - \Big[\Big(\sum_{j \neq i} v_j(\alpha(\theta), \theta_j)\Big) - c(\alpha(\theta))\Big] + c_i(\alpha(0, \theta_{-i})) \\ &= \Big[\Big(\sum_{j \neq i} v_j(\alpha(0, \theta_{-i}), \theta_j)\Big) - c(\alpha(0, \theta_{-i}))\Big] - \Big[\Big(\sum_{j \neq i} v_j(\alpha(\theta), \theta_j)\Big) - c(\alpha(\theta))\Big] + c_i(\alpha(0, \theta_{-i})).\end{aligned}$$

The cost-sharing pivotal mechanism simply adds $c_i(\alpha(0, \theta_{-i}))$—the cost of what would have been chosen without i—to i's transfer in the pivotal mechanism. The pivotal mechanism charges everyone zero and additionally charges those who are pivotal the welfare loss they impose on others. The cost-sharing pivotal mechanism, on the other hand, charges everyone their fair share of the cost of what would have been chosen without them and additionally charges those who are pivotal the welfare loss they impose on others. Notice that if no individual is pivotal, the cost-sharing pivotal mechanism is *ex-post budget-balanced (EPBB)*—i.e., $\sum_{i=1}^{n} \tau_i(\theta) = c(\alpha(\theta))$.

While the pivotal mechanism can be characterized as the revenue-maximizing Groves mechanism subject to universal participation, the cost-sharing pivotal mechanism can be characterized as the revenue-maximizing Groves mechanism subject to cost-sharing universal participation.

Theorem 1. *Given any environment $\mathcal{E}_X \in \mathbb{E}_X$, a mechanism maximizes ex-post revenue among all mechanisms which satisfy strategy-proofness, efficiency, and cost-sharing universal participation if and only if it is a cost-sharing pivotal mechanism.*

Proof. Given any environment $\mathcal{E}_X \in \mathbb{E}_X$, a mechanism f is strategy-proof and efficient if and only if it is a Groves mechanism by Holmström (1979). Let $f = (\alpha, \tau)$ be a Groves mechanism with $\tau_i(\theta) = g_i(\theta_{-i}) - [\sum_{j \neq i} v_j(\alpha(\theta), \theta_j) - c(\alpha(\theta))]$ for some $g_i : \Theta_{-i} \to \mathbb{R}$. We would like to construct $g_i(\theta_{-i})$ to maximize revenue subject to cost-sharing universal participation. In particular,

$$g_i(\theta_{-i}) = \inf_{\theta_i \in \Theta_i} \; v_i(\alpha(\theta), \theta_i) + \sum_{j \neq i} v_j(\alpha(\theta), \theta_j) - c(\alpha(\theta)) - v_i(\alpha(0, \theta_{-i}), \theta_i) + c_i(\alpha(0, \theta_{-i})).$$

Plugging $\theta_i = 0$ into the objective function, we have $\sum_{j \neq i} v_j(\alpha(0, \theta_{-i}), \theta_j) - c(\alpha(0, \theta_{-i})) + c_i(\alpha(0, \theta_{-i}))$, which is indeed the minimum since by definition of α, for all θ_i,

$$v_i(\alpha(\theta), \theta_i) + \sum_{j \neq i} v_j(\alpha(\theta), \theta_j) - c(\alpha(\theta)) \geq v_i(\alpha(0, \theta_{-i}), \theta_i) + \sum_{j \neq i} v_j(\alpha(0, \theta_{-i}), \theta_j) - c(\alpha(0, \theta_{-i})),$$

which holds if and only if for all θ_i,

$$\begin{aligned}&v_i(\alpha(\theta), \theta_i) + \sum_{j \neq i} v_j(\alpha(\theta), \theta_j) - c(\alpha(\theta)) - v_i(\alpha(0, \theta_{-i}), \theta_i) + c_i(\alpha(0, \theta_{-i})) \\ &\qquad \geq \sum_{j \neq i} v_j(\alpha(0, \theta_{-i}), \theta_j) - c(\alpha(0, \theta_{-i})) + c_i(\alpha(0, \theta_{-i})).\end{aligned}$$

Hence, $g_i(\theta_{-i}) = [\sum_{j \neq i} v_j(\alpha(0, \theta_{-i}), \theta_j) - c(\alpha(0, \theta_{-i}))] + c_i(\alpha(0, \theta_{-i}))$ as desired. ■

6 Convex Environments with Net Values

The standard approach in the literature to modeling implementation costs is to embed them into the individuals' values. Instead of using an individual i's value v_i for some alternative y, say constructing a park, we use instead an individual i's net value $\tilde{v}_i$ for the construction of the park *and* paying a tax equal to her fair share of its cost $c_i(y)$. In particular, if i has type θ_i, her *net value* for y, including a tax of $c_i(y)$, is $\tilde{v}_i(y, \theta_i) = v_i(y, \theta_i) - c_i(y)$, and her *net transfer*—the additional payment beyond the tax $c_i(\alpha(\theta))$—is $\tilde{\tau}_i(\theta) = \tau_i(\theta) - c_i(\alpha(\theta))$.

Mechanically, this means that when facing an environment with implementation costs, we may simply specify the taxes for each alternative, ask individuals to report their values net of this tax, and utilize these values as if they were the individuals' actual values and there were no implementation costs. I call this the *net value approach.* Indeed, it is because of this approach that it is common in the literature not to model implementation costs explicitly.[30]

Using net values, we may rewrite the set of efficient decisions by

$$A^*(\theta) = \arg\max_{y \in Y} \sum_{i \in I} \Big(v_i(y, \theta_i) \Big) - c(y) = \arg\max_{y \in Y} \sum_{i \in I} \tilde{v}_i(y, \theta_i).$$

This technique—interpreting reported values as net values and disregarding implementation costs; or equivalently, replacing v_i with $\tilde{v}_i$ and setting $c_i(y) = c(y) = 0$ for all y and i—produces equivalent statements for several definitions including strategy-proofness, efficiency, individual rationality, and the Groves mechanisms. Importantly, it does not produce equivalent statements for universal participation and the pivotal mechanism.

This is because the "zero" value changes meaning when we replace values with net values. Recall that $0 \in \Theta_i$ is a type for which $v_i(y, 0) = 0$ for all y—i.e., i's value for each alternative is zero. Define $\tilde{0} \in \Theta_i$ to be a type for which $\tilde{v}_i(y, \tilde{0}) = 0$ for all y—i.e., i's *net* value for each alternative is zero. Given type $\tilde{0}$, i's value for y is then $v_i(y, \tilde{0}) = \tilde{v}_i(y, \tilde{0}) + c_i(y) = c_i(y)$. In other words, if i has a net value for y of zero, she in fact has a positive value for y itself—a value exactly equal to her fair share of its cost.

Plugging net values into universal participation produces the following distinct constraint, which I call net-value universal participation.

[30] See Section 2.1 for further discussion.

Definition 2.1. A mechanism $f = (\alpha, \tau)$ satisfies *net-value universal participation* if for all θ and i,

$$\tilde{v}_i(\alpha(\theta), \theta_i) - \tilde{\tau}_i(\theta) \geq \tilde{v}_i(\alpha(\tilde{0}, \theta_{-i}), \theta_i),$$

or equivalently,

$$v_i(\alpha(\theta), \theta_i)) - \tau_i(\theta)) \geq v_i(\alpha(\tilde{0}, \theta_{-i}), \theta_i) - c_i(\alpha(\tilde{0}, \theta_{-i})).$$

The first line makes clear that net-value universal participation is simply universal participation with net values. The second line expresses this condition in terms of actual values v_i and transfers τ_i, making clear what it states about the underlying fundamentals: each individual should always prefer to participate in the mechanism rather than to not participate—that is, rather than to receive the alternative chosen had she valued each good at precisely her fair share of its cost and to be taxed her fair share of the cost of that good.

Similarly, plugging net values into and removing implementation costs from the pivotal mechanism produces the following distinct mechanism, known as the Clarke mechanism.

Definition 2.2. A mechanism $f = (\alpha, \tau)$ is a *Clarke mechanism* if the decision rule is efficient and the net transfer rule satisfies

$$\begin{aligned}\tilde{\tau}_i(\theta) &= \Big[\max_{y \in Y}\Big(\sum_{j \neq i} \tilde{v}_j(y, \theta_j)\Big)\Big] - \Big[\Big(\sum_{j \neq i} \tilde{v}_j(\alpha(\theta), \theta_j)\Big)\Big] \\ &= \Big[\sum_{j \neq i} \tilde{v}_j(\alpha(\tilde{0}, \theta_{-i}), \theta_j)\Big] - \Big[\sum_{j \neq i} \tilde{v}_j(\alpha(\theta), \theta_j)\Big],\end{aligned}$$

or equivalently, if the transfer rule satisfies

$$\begin{aligned}\tau_i(\theta) &= \Big[\max_{y \in Y}\Big(\sum_{j \neq i} v_j(y, \theta_j) - c_j(y)\Big)\Big] - \Big[\Big(\sum_{j \neq i} v_j(\alpha(\theta), \theta_j) - c_j(\alpha(\theta))\Big)\Big] + c_i(\alpha(\theta)) \\ &= \Big[\Big(\sum_{j \neq i} v_j(\alpha(\tilde{0}, \theta_{-i}), \theta_j) - c_j(\alpha(\tilde{0}, \theta_{-i}))\Big)\Big] - \Big[\Big(\sum_{j \neq i} v_j(\alpha(\theta), \theta_j) - c_j(\alpha(\theta))\Big)\Big] + c_i(\alpha(\theta)).\end{aligned}$$

As before, the first equation makes clear that the Clarke mechanism is simply a pivotal mechanism with net values and no implementation costs. The second equation expresses actual transfers τ_i in terms of actual values v_i. The Clarke mechanism is one way, and until now the only proposed way, to generalize the pivotal mechanism to environments with implementation costs. The cost-sharing pivotal mechanism, defined in Section 5, is another way. The Clarke mechanism inserts net values into and removes implementation costs from i's transfer in the pivotal mechanism. The cost-sharing pivotal mechanism adds $c_i(\alpha(0, \theta_{-i}))$—the cost of what would have been chosen without i—to i's transfer in the pivotal mechanism. These are two distinct mechanisms with markedly different interpretations and properties satisfied. See Sections 2.3 and 2.4 and, in particular, Figures 1 and 2 for a detailed comparison of the two.

Reinterpreting the result from Fryxell (2023) in Section 5 with net values immediately implies that the Clarke mechanism can be characterized as the revenue-maximizing Groves mechanism subject to net-value universal participation.

Corollary (Fryxell, 2023). *Given any environment $\mathcal{E}_X \in \mathbb{E}_X$, a mechanism maximizes ex-post revenue among all mechanisms which satisfy strategy-proofness, efficiency, and net-value universal participation if and only if it is a Clarke mechanism.*

Although cost-sharing universal participation neither implies nor is implied by net-value universal participation, the Clarke mechanism always raises more revenue than the cost-sharing pivotal mechanism.

Proposition 1. *Given any environment $\mathcal{E}_X \in \mathbb{E}_X$, the transfer for each individual i in a Clarke mechanism is no less than that in a cost-sharing pivotal mechanism.*

Proof. $\tau_i^{\text{Clarke}}(\theta) \geq \tau_i^{\text{CSP}}(\theta)$ if and only if

$$\Big[\max_{y\in Y}\Big(\sum_{j\neq i} v_j(y,\theta_j)\Big)-\Big(c(y)-c_i(y)\Big)\Big] \geq \Big[\Big(\sum_{j\neq i} v_j(\alpha(0,\theta_{-i}),\theta_j)\Big)-c(\alpha(0,\theta_{-i}))\Big]+c_i(\alpha(0,\theta_{-i})).$$

■

In fact, one of the main selling points of the Clarke mechanism is that it never runs a budget deficit.

Definition 2.3. A mechanism $f = (\alpha, \tau)$ satisfies *no-deficit (ND)* if it never runs a budget deficit—i.e., $\sum_{i=1}^{n} \tau_i(\theta) \geq c(\alpha(\theta))$ for all θ.

Be that as it may, the fuzzy taxation principle argues that satisfying no-deficit is a red herring, and what we should really care about is that deviations in either direction from budget-balance are small (see Section 2.4). Moreover, Theorem 2 shows that no-deficit cannot be satisfied alongside a weak fairness criterion I call *no-extortion.* No-extortion is a minimal standard of fairness in public good environments. It says that if nothing is produced, no payments are collected.

Definition 2.4. A mechanism $f = (\alpha, \tau)$ satisfies *no-extortion (NE)* if when nothing is produced, no payments are collected. That is, $\alpha(\theta) = 0$ implies $\tau_i(\theta) \leq 0$.[31]

[31] One might think it is perfectly reasonable to charge a payment to an individual who sways the decision towards something they prefer. With negative values, it is possible to sway the decision

The Clarke mechanism violates no-extortion (see Example 1). The following definition and proposition are useful to prove Theorem 2. A potential project-starter is an individual for whom the project may or may not be produced, depending on her report.

Definition 2.5. A player i is a *potential project-starter for* θ_{-i} if $\alpha(0, \theta_{-i}) = 0$ and there exists $\theta_i \in \Theta_i$ such that $\alpha(\theta) > 0$.

A mechanism satisfying strategy-proofness, efficiency, and no-extortion can charge potential project-starters no more than their pivotal payment.

Proposition 2. *Given any environment $\mathcal{E}_X \in \mathbb{E}_X$, a mechanism satisfies strategy-proofness, efficiency, and no-extortion if and only if it charges no more than a pivotal payment for potential project-starters. That is, if player i is a potential project-starter for θ_{-i}, then*

$$\tau_i(\theta) \leq \Big[\Big(\sum_{j\neq i} v_j(\alpha(0,\theta_{-i}),\theta_j)\Big) - c(\alpha(0,\theta_{-i}))\Big] - \Big[\Big(\sum_{j\neq i} v_j(\alpha(\theta),\theta_j)\Big) - c(\alpha(\theta))\Big].$$

Proof. Given any environment $\mathcal{E}_X \in \mathbb{E}_X$, a mechanism f is strategy-proof and efficient if and only if it is a Groves mechanism by Holmström (1979). Let $f = (\alpha, \tau)$ be a Groves mechanism with $\tau_i(\theta) = [\sum_{j\neq i} v_j(\alpha(0,\theta_{-i}),\theta_j) - c(\alpha(0,\theta_{-i}))] - [\sum_{j\neq i} v_j(\alpha(\theta),\theta_j) - c(\alpha(\theta))] + h_i(\theta_{-i})$ for some $h_i : \Theta_{-i} \to \mathbb{R}$. If i is a potential project-starter for θ_{-i}, then no-extortion requires

$$\tau_i(0,\theta_{-i}) = \Big[\Big(\sum_{j\neq i} v_j(0,\theta_j)\Big) - c(0)\Big] - \Big[\Big(\sum_{j\neq i} v_j(0,\theta_j)\Big) - c(0)\Big] + h_i(\theta_{-i}) = h_i(\theta_{-i}) \leq 0.$$

■

A strategy-proof and efficient mechanism cannot satisfy both no-extortion and no-deficit. The Clarke mechanism satisfies strategy-proofness, efficiency, and no-deficit and hence violates no-extortion.

Theorem 2. *There does not exist any mechanism that satisfies strategy-proofness, efficiency, no-extortion, and no-deficit in all binary public good environments*[32] *(and hence also in all finite, continuous, and convex public good environments).*

from constructing the good to not. One could then define no-extortion as follows. A mechanism satisfies *no-extortion* if, for any individual i with type θ_i such that $v_i(y, \theta_i) \geq v_i(0, \theta_i)$ for all y, $\alpha(\theta) = 0$ implies $\tau_i(\theta) \leq 0$. In this paper, I primarily consider the case of non-negative values. Hence, for simplicity and elegance I define no-extortion without this prerequisite. It is immediate that all results in this paper hold identically for either definition.

[32]A binary public good environment has two alternatives (the public good and the status quo) and non-negative values for the public good. In particular, it is a finite public good environment, as in Section 8, with $K = 1$.

Proof. Consider a binary public good environment with three players each with value 3 for the public good which costs 8. The efficient decision is to produce the public good. Each player is a potential project-starter, so in order to satisfy no-extortion their transfer must be no larger than their pivotal payment by Proposition 2, which is 2. Hence, total revenue can be no more than 6, violating no-deficit. ■

7 Continuous Public Good Environments

Let $\mathbb{E}_C \subset \mathbb{E}_X$ be the set of environments with unidimensional social alternatives; quasilinear preferences; private, absolutely continuous, non-decreasing, and convex values; and a fully indifferent type. In particular, $\mathbb{E}_C$ is the set of environments in $\mathbb{E}_X$ where each of the following are true. Social alternatives are unidimensional: $Y = \mathbb{R}_+$. For each i and θ_i, v_i is absolutely continuous and non-decreasing in y with $v_i(0, \theta_i)$ normalized to 0 for all θ_i.[33] Costs c and fair shares c_i for each i are absolutely continuous and non-decreasing with $c(0)$ normalized to 0.[34]

Individual rationality is a common desiderata in the mechanism design literature.

Definition 3.1. A mechanism $f = (\alpha, \tau)$ is *individually-rational* if for all i, θ,

$$\tau_i(\theta) \leq v_i(\alpha(\theta), \theta_i) = \int_0^{\alpha(\theta)} v_i'(y, \theta_i) \text{ d}y.$$

Generally, individual rationality is interpreted as a participation constraint, in which case it can be understood to say that an individual should always prefer to participate in the mechanism rather than to not participate, consume nothing, and pay nothing. This is not appropriate in public good environments (see Sections 2.2 and 5). We may also interpret individual rationality as a fairness principle, in which case it can be understood to say that an individual's payment should never be larger than her value for the chosen alternative. As discussed in Section 2.2, this notion of fairness is too demanding for public good environments. In these environments, fairness should capture a sense of community, wherein each individual has, and understands they have, a responsibility to play their part. The fair pricing principle captures this sentiment.

Definition 3.2. A mechanism $f = (\alpha, \tau)$ satisfies the *fair pricing principle (FPP)* if for all i, θ,

$$\tau_i(\theta) \leq \max\Big\{v_i(\alpha(\theta), \theta_i),\ c_i(\alpha(\theta))\Big\} = \max\Big\{\int_0^{\alpha(\theta)} v_i'(y, \theta_i) \text{ d}y,\ \int_0^{\alpha(\theta)} c_i'(y) \text{ d}y\Big\}.$$

[33] Hence, $v_i(y, \theta_i) = \int_0^y v_i'(z, \theta_i) \text{ d}z$ for all i.
[34] Hence, $c(y) = \int_0^y c'(z) \text{ d}z$ and $c_i(y) = \int_0^y c_i'(z) \text{ d}z$ for all i.

The fair pricing principle says that an individual's payment should not be larger than the maximum of her value for the chosen alternative and her fair share of its cost. In other words, it is fair to ask an individual to pay up to her fair share of the cost of what is produced, and it is also fair to ask an individual to pay more as long as this amount is less than her value for what is produced. The fair pricing principle per unit captures the same sentiment unit by unit.

Definition 3.3. A mechanism $f = (\alpha, \tau)$ satisfies the *fair pricing principle per unit (FPP per unit)* if for all i, θ,

$$\tau_i(\theta) \leq \int_0^{\alpha(\theta)} \max\left\{v_i'(y, \theta_i),\, c_i'(y)\right\} \mathrm{d}y.$$

The fair pricing principle per unit says that for each unit of the public good, it is fair for an individual to pay up to the maximum of her marginal value for that unit and her fair share of its marginal cost. Since the maximum is taken unit by unit, this is a strictly weaker condition than the fair pricing principle—allowing, for instance, that an individual be taxed her fair share for the first marginal unit of the good and then be taxed her marginal value for the second, which may total more than the maximum of her fair share and her value for both units of the good (see Example 3).

It is clear that individual rationality implies the fair pricing principle which implies the fair pricing principle per unit which implies no-extortion. One may wonder if cost-sharing universal participation also implies no-extortion. The answer is yes, but with a minor qualification. This qualification turns out to be important for fairness in general. I discuss it now.

In a continuous public good environment, it is intuitive that if i's marginal value increases everywhere, all else fixed, an efficient decision rule would never select a strictly smaller amount of the public good. This intuition is nearly correct, but for a technical reason not entirely so. It is true if the set of efficient decisions is always a singleton, but it is not true in general. What is true is that increasing an individual's marginal value increases the set of efficient decisions in the strong set order.[35]

Proposition 3. *Given any environment $\mathcal{E}_C \in \mathbb{E}_C$, if $v_i'(y, \hat{\theta}_i) \geq v_i'(y, \theta_i)$ for all i, y, and θ_i, then $A^*(\hat{\theta}) \geq A^*(\theta)$ in the strong set order.*

Proof. For any θ, $y^* \in A^*(\theta)$ if and only if for any $y' \leq y^* \leq y''$,

$$\int_{y'}^{y^*} \sum_{i \in I} v_i'(y, \theta_i) - c'(y) \ \mathrm{d}y \geq 0 \geq \int_{y^*}^{y''} \sum_{i \in I} v_i'(y, \theta_i) - c'(y) \ \mathrm{d}y.$$

[35] For any $A, B \subseteq \mathbb{R}$, $A \leq B$ in the *strong set order* if for any $a \in A$ and $b \in B$, $\min\{a, b\} \in A$ and $\max\{a, b\} \in B$. See Milgrom and Shannon (1994).

That is, y^* is efficient if and only if moving from any $y' \leq y^*$ to y^* weakly increases welfare and moving from y^* to any $y'' \geq y^*$ weakly decreases welfare.

Suppose $v_i'(y, \hat{\theta}_i) \geq v_i'(y, \theta_i)$ for all i, y, and θ_i, $y^* \in A^*(\theta)$, $\hat{y}^* \in A^*(\hat{\theta})$, and $\hat{y}^* \leq y^*$. Then

$$\int_{\hat{y}^*}^{y^*} \sum_{i \in I} v_i'(y, \theta_i) - c'(y) \, \mathrm{d}y \geq 0 \geq \int_{\hat{y}^*}^{y^*} \sum_{i \in I} v_i'(y, \hat{\theta}_i) - c'(y) \, \mathrm{d}y \geq \int_{\hat{y}^*}^{y^*} \sum_{i \in I} v_i'(y, \theta_i) - c'(y) \, \mathrm{d}y$$

where the first inequality follows since $y^* \in A^*(\theta)$, the second since $\hat{y}^* \in A^*(\hat{\theta})$, and the third since $v_i'(y, \hat{\theta}_i) \geq v_i'(y, \theta_i)$ for all i, y, and θ_i. Hence, $y^* \in A^*(\hat{\theta})$ and $\hat{y}^* \in A^*(\theta)$. ■

Because an efficient decision rule can select *any* efficient decision, nothing prevents it from selecting a strictly smaller amount of the public good when marginal values increase, provided this remains in the set of efficient decisions. A trivial example is if an increase in i's marginal value doesn't change the set of efficient decisions. In such a case, an efficient decision rule may very well select a strictly smaller alternative. It turns out that such decision rules will cause problems for fairness, in the sense of violating no-extortion or any of the stronger notions above (see Example 2). This motivates the following definition, which aligns efficient decision rules with our intuition.

Definition 3.4. A decision rule $\alpha : \Theta \to \mathbb{R}_+$ is *monotone* if $v_i'(y, \hat{\theta}_i) \geq v_i'(y, \theta_i)$ for all $y \geq 0$ and $i \in I$ implies $\alpha(\hat{\theta}) \geq \alpha(\theta)$.

A decision rule is monotone if increasing any individual's value function pointwise never decreases the decision. With a non-monotone decision rule, the cost-sharing pivotal mechanism violates no-extortion.

Example 2 (With a non-monotone decision rule, the CSP violates NE.)**.** Let $I = \{i, j\}$ and $Y = \{0, 1, 2\}$, representing no park, a small park, and a large park, respectively. The cost of the small park is 10 and the cost of the large park is 20, and each individual's fair share is 5 for the small park and 10 for the large park. Individual i values the small park at 0 and the large park at 2, and individual j values the small park at 10 and the large park at 15. Given this, the set of efficient decisions is $\{0, 1\}$. Both no park and the small park result in a total welfare of zero, which is optimal. Suppose α selects 0—no park.

Ignoring i's preferences (had she had zero value for both the small and large park), the set of efficient decisions remains unchanged. Suppose α is non-monotonic and for this type profile selects 1—the small park. The CSP charges i the welfare loss she imposes on others by having her preferences taken into account (0, since i's opponents' total welfare is zero with and without the small park) plus her fair share of the cost of what would have been produced without her (5, her fair share of the small park), violating no-extortion. □

With a monotone decision rule, cost-sharing universal participation implies no-extortion.

Proposition 4. *Given any environment $\mathcal{E}_C \in \mathbb{E}_C$, if the decision rule is monotone, cost-sharing universal participation implies no-extortion.*

Proof. Suppose $\alpha(\theta) = 0$. Since α is monotone, $\alpha(0, \theta_{-i}) = 0$. Then CS-UP implies $\tau_i(\theta) \leq 0$ as desired. ■

With a monotone decision rule, the cost-sharing pivotal mechanism satisfies no-extortion and the fair pricing principle per unit, but it violates the fair pricing principle (see Example 3). That said, the cost-sharing pivotal mechanism satisfies the fair pricing principle under the natural assumption that marginal values are non-increasing and marginal costs are non-decreasing.

Assumption 1. Suppose $v_i'(y, \theta_i)$ is non-increasing in y for all i and θ_i and $c'(y)$ and $c_i'(y)$ are non-decreasing for all i.

Example 3 (Without Assumption 1, the CSP violates the FPP)**.** Let $I = \{i, j\}$ and $Y = \{0, 1, 2\}$, representing no park, a small park, and a large park, respectively. The cost of the small park is 10 and the cost of the large park is 20, and each individual's fair share is 5 for the small park and 10 for the large park. Individual i values the small park at 0 and the large park at 11 (violating Assumption 1), and individual j values the small park at 11 and the large park at 11. Total welfare is 0 with no park, $0 + 11 - 10 = 1$ with the small park, and $11 + 11 - 20 = 2$ with the large park, so the large park is the efficient decision.

Ignoring i's preferences, the small park would be produced for a total welfare to i's opponents of 1. With i, the large park is produced for a total welfare to i's opponents of $11 - 20 = -9$. The CSP charges i the welfare loss she imposes on others by having her preferences taken into account $(1 - (-9) = 10)$ plus her fair share of the cost of what would have been produced without her (5), for a total of 15. The FPP requires that i pay no more than the maximum of her value and her fair share of what is produced. Since 15 is larger than both i's value for the large park (11) and her fair share of its cost (10), the CSP violates the FPP.[36] □

In Theorem 1, we showed that the cost-sharing pivotal mechanism can be characterized as the revenue-maximizing Groves mechanism subject to cost-sharing universal participation. We now show that the cost-sharing pivotal mechanism can also be

[36] On the other hand, the FPP per unit requires that i pay no more than the maximum of her marginal value and her marginal fair share for each marginal unit of what is produced—i.e., she can pay up to 5 for the first unit and up to 11 for the second (up to 16 total).

characterized as the revenue-maximizing Groves mechanism subject to the fair pricing principle.

Theorem 3. *Consider any environment $\mathcal{E}_C \in \mathbb{E}_C$ and any efficient and monotone decision rule α. A mechanism $f = (\alpha, \tau)$ maximizes ex-post revenue among all mechanisms which satisfy strategy-proofness, efficiency, and the fair pricing principle per unit if and only if it is a cost-sharing pivotal mechanism. Under Assumption 1, the same holds replacing the fair pricing principle per unit with the fair pricing principle.*

Proof Commentary. The proof proceeds as in Theorem 1, though with several more cases. □

Note that in binary public good environments,[37] where a public project can either be provided or not, Assumption 1 always holds.

Stepping back, the cost-sharing pivotal mechanism satisfies cost-sharing universal participation and, with a monotone decision rule, no-extortion, the fair pricing principle per unit, and, under Assumption 1, the fair pricing principle. On the other hand, the Clarke mechanism violates no-extortion, the fair pricing principle per unit, the fair pricing principle, and, with a monotone decision rule, cost-sharing universal participation.

While we have characterized the cost-sharing pivotal mechanism as the mechanism which maximizes ex-post revenue subject to strategy-proofness, efficiency, and either cost-sharing universal participation or the fair pricing principle, our goal is not actually to maximize revenue. These simply turn out to be two characterizing features of the mechanism. Our goal is to balance the budget, or to come as close as possible to doing so. It is then important to ask the question: does the cost-sharing pivotal mechanism raise *too much* revenue? In particular, does there exist a mechanism that always comes closer to ex-post budget-balance than the cost-sharing pivotal mechanism? The answer is no.

Theorem 4. *Consider any environment $\mathcal{E}_C \in \mathbb{E}_C$. Suppose that for any i and θ_{-i}, $\max A^*(0, \theta_{-i})$ exists and there exists $\theta_i \in \Theta_i$ such that*

$$v_i'(y, \theta_i) = \begin{cases} \phi_i(y) & \textit{if } y \leq \max A^*(0, \theta_{-i}) \\ 0 & \textit{if } y > \max A^*(0, \theta_{-i}) \end{cases}, \tag{1}$$

where $\phi_i(y) \geq \max_{j \neq i} v_j'(y, \theta_j)$ and $\phi_i(y) > 0$ for all y.[38] *For any efficient and monotone decision rule α, there is no strategy-proof and cost-sharing universal participation*

[37]Binary public good environments are a special case of finite public good environments, which are a special case of continuous public good environments. See Section 8.

[38]This is a weak richness condition on the domain of values which says that each individual may value marginal units of the public good as much as any other individual and that this marginal value may drop to zero at any point.

mechanism that is no farther from ex-post budget-balance than a cost-sharing pivotal mechanism for every θ. The same holds replacing cost-sharing universal participation with the fair pricing principle per unit. Under Assumption 1, the same holds replacing the fair pricing principle per unit with the fair pricing principle.

Proof Sketch. For any θ_{-i}, if θ_i satisfies (1) then every agent's pivotal payment is zero, so the CSP is either exactly budget-balanced or runs a deficit at θ.[39] By Holmström (1979), an efficient mechanism is strategy-proof if and only if i's transfer can be expressed as the sum of her pivotal payment and a term that depends only on her opponents' reports $h_i(\theta_{-i})$. If $h_i(\theta_{-i}) < c_i(\alpha(0, \theta_{-i}))$ the mechanism runs a strictly larger budget-deficit at θ than a CSP, and if $h_i(\theta_{-i}) > c_i(\alpha(0, \theta_{-i}))$ the mechanism violates cost-sharing universal participation by Theorem 1 and the fair pricing principle by Theorem 3. ■

8 Finite Public Good Environments

In this section, I consider random sequences of finite public good environments with increasing population size n and show that the cost-sharing pivotal mechanism is asymptotically ex-post budget-balanced (Theorem 5) and that it is effectively the unique such mechanism which also satisfies strategy-proofness, efficiency, and at least one of cost-sharing universal participation and the fair pricing principle (Theorem 6). Three key facets of this analysis are the following.

1. I allow the cost of the public goods to vary arbitrarily with n.
2. I allow individual cost shares to depend arbitrarily on observable characteristics and for observable characteristics to be arbitrarily correlated with individual types.
3. The net value approach cannot accommodate either.

I discuss each in turn and then present the formal analysis.

8.1 The cost of the public good can vary arbitrarily with population size

It is common that asymptotic results involving sequences of public good environments depend on how the sequence of cost functions grows with n.[40] I allow costs to vary arbitrarily with n. This is an important distinction. To see this, consider for simplicity a binary public good environment. We do not know the population distribution of values. We would like to talk about what happens in large populations for any such distribution. That is, we would like to learn about what happens as we increase the

[39] If the efficient decision is unique for each θ, this would read simply as: for any θ_{-i}, if θ_i satisfies (1) then no agent is pivotal, so the CSP is EPBB at θ.

[40] See e.g., Mailath and Postlewaite (1990) and Xi and Xie (2021).

number of i.i.d. draws from any distribution holding all else constant. But what does it mean to hold all else constant? In particular, what about the relationship between the population size and the cost of the public good should be held constant?

If we assume costs grow very slowly or not at all, then as the population grows large, the efficient decision is almost surely to produce the good, making the public good provision problem trivial, no matter the underlying population distribution. If it is efficient to produce the good with arbitrarily high probability, we do not need to elicit preferences—we can simply produce the good and charge everyone their fair share. Similarly, if we assume costs grow sufficiently quickly, then as the population grows large, the efficient decision is almost surely *not* to produce the good, again making the problem trivial. To avoid this situation in which the chosen sequence of cost functions plays a significant (and artificial) role in the analysis, we would like our results to be robust to *any* such sequence—and indeed they are.[41]

8.2 Individual cost shares can vary arbitrarily with observables

So far the arbitrariness of (exogenously given) cost shares has not played a role in the analysis. But this will no longer be the case, as we will need to make statements about deviations from budget-balance, and this depends on the individuals' cost shares—and, in particular, on how they correlate with the individuals' types. We can make probabilistic statements about outcomes with just a prior on Θ—the environment itself need not be random. However, since cost shares may be a function of individuals' observable characteristics (see Section 4), which may be correlated with types, we must now consider environments which are themselves random. Notably, when making probabilistic statements about outcomes, the commonly assumed special case of equal cost shares, $c_i(y) = c(y)/n$ for all i, assumes away an important dimension of fair public good provision—that fair shares may depend on observables, and observables may be correlated with types.

To gain some intuition, consider a CSP with equal cost shares. It might be that the expected distance from ex-post budget-balance is small because almost everyone is not pivotal and so pays their equal share, which gets increasingly close to the full cost. However, now suppose an individual's fair share is determined by her income, which is highly correlated with her value for the public good. In particular, suppose that those who tend to be pivotal tend to have an extremely large fair share, leaving much smaller fair shares for everyone else. Now, even if almost everyone is not pivotal, they

[41] Arguably, the object that should be held constant is precisely the probability that the efficient decision is to produce the good. That is, in any finite public good provision environment, we would like $\mathbb{P}(\alpha^n(\theta^n) = y)$ to be constant in n for each y. Computing sequences of cost functions with this property is not trivial. Luckily, we do not have to worry about this—since the results hold for any sequence $\{c^n(\cdot)\}_{n\in\mathbb{N}}$, they clearly hold for those sequences as well.

will pay a much smaller amount, which may not approach the full cost. In particular, the CSP might be asymptotically ex-post budget-balanced in the former case but not the latter.

Encouragingly, the results herein are robust to all the aforementioned concerns. Theorem 5 shows that the CSP is asymptotically ex-post budget-balanced no matter how the cost of the public goods vary with n, no matter how fair shares are constructed from observables, and no matter how types and observables are correlated.

8.3 The net value approach cannot accommodate these results

Importantly, the net value approach cannot accommodate these results. Consider again a binary public good environment. We would like to fix a population distribution of values and consider sequences of i.i.d. random draws from this distribution. But while individuals' *values* are i.i.d., individuals' *net values* depend on their cost share, which depends on the total cost of the public good and the observable characteristics of the others, and hence are in general not i.i.d. Indeed, assuming net values are i.i.d. requires the underlying assumptions that 1) cost shares are split equally across individuals and 2) costs increase linearly with population size.

8.4 Formal analysis

I now introduce these objects formally. Let $\mathbb{E}_F \subset \mathbb{E}_X$ be the set of environments with ordered and finite social alternatives; quasilinear preferences; private, non-decreasing, and convex values; and a fully indifferent type. In particular, $\mathbb{E}_F$ is the set of environments in $\mathbb{E}_X$ where each of the following are true. Social alternatives are finite and ordered: $Y = \{0, 1, \ldots, K\}$ for some $K \in \mathbb{N}$. For each i and θ_i, v_i is non-decreasing in y with $v_i(0, \theta_i)$ normalized to 0 for all θ_i. For simplicity, let $v_i(y, \theta_i) = \theta_i(y)$ for all $i \in I$ and $y = 1, \ldots, K$, where $\theta_i(y)$ is the yth component of θ_i. Costs c and fair shares c_i for each i are non-decreasing with $c(0)$ normalized to 0. Note that every finite environment $\mathcal{E}_F \in \mathbb{E}_F$ can be equivalently represented by a continuous environment $\mathcal{E}_C \in \mathbb{E}_C$ in which we restrict attention to decision rules α with range $Y = \{0, 1, \ldots, K\}$ and v_i, c, and c_i for all i are piecewise linear on $\{[0, 1), [1, 2), \ldots, [K-1, K), [K, \infty)\}$ and constant on $[K, \infty)$. In this sense, $\mathbb{E}_F \subset \mathbb{E}_C$.

Let $Y_0 = \{0, \ldots, K\}$ be a set of social alternatives, $\Theta_0 \subseteq \mathbb{R}^K_+$ be a set of types, and Z_0 be an arbitrary set of observable characteristics. Consider any distribution over $\Theta_0 \times Z_0$ such that the variance of an individual's value differences is finite—i.e., $\mathbb{E}\big((\theta_i(b) - \theta_i(a))^2\big) < \infty$ for all $a, b \in Y$. Let $\Delta(\Theta_0 \times Z_0)$ be the set of all such distributions. Let $(\theta_i, z_i)_{i \in \mathbb{N}}$ be a sequence of i.i.d. random vectors with distribution $F \in \Delta(\Theta_0 \times Z_0)$.

For all $n \in \mathbb{N}$, let $I^n = \{1, \ldots, n\}$, $Y^n = Y_0$, $\Theta^n = (\Theta_0)^n$, $Z^n = (Z_0)^n$, $v_i^n(y, \theta_i) = \theta_i(y)$

for all $i \in I^n$ and $y = 1, \ldots, K$, $c^n : Y^n \to \mathbb{R}_+$ be any non-decreasing function, and $\phi^n : Y^n \times Z^n \to \mathbb{R}^n$ be any function such that $\sum_{i=1}^n \phi_i^n(y, z) = c(y)$ for all $y \in Y^n$ and $z \in Z^n$.

Let $\theta^n = (\theta_i)_{i=1}^n$ denote the sequence of types up to n and $z^n = (z_i)_{i=1}^n$ the sequence of observable characteristics up to n. For each n, the realization of z^n determines the cost shares for each individual $c_i^n(y) = \phi_i^n(y, z)$, and given a sequence of mechanisms $((\alpha^n, \tau^n))_{n\in\mathbb{N}}$, the realization of θ^n determines the public good provision level $\alpha^n(\theta^n)$ and, along with z^n, each individual's transfer $\tau_i^n(\theta^n)$. Let $(\theta^n, z^n, \mathcal{E}^n)_{n\in\mathbb{N}}$ denote a sequence of finite environments generated by i.i.d. draws from some $F \in \Delta(\Theta_0 \times Z_0)$.

Definition 5.1. Given any sequence of finite environments $(\theta^n, z^n, \mathcal{E}^n)_{n\in\mathbb{N}}$ generated by i.i.d. draws from some $F \in \Delta(\Theta_0 \times Z_0)$, a sequence of mechanisms $((\alpha^n, \tau^n))_{n\in\mathbb{N}}$ is *asymptotically ex-post budget-balanced (AEPBB)* if

1. the probability of ex-post budget-balance goes to one as n goes to infinity, i.e.,

$$\mathbb{P}\Big(\sum_{i=1}^n \tau_i^n(\theta^n) - c^n(\alpha^n(\theta^n)) = 0\Big) \to 1 \quad \text{as} \quad n \to \infty,$$

   and

2. the expected distance from ex-post budget-balance per capita goes to zero as n goes to infinity, i.e.,

$$\frac{1}{n}\mathbb{E}\Big(\Big|\sum_{i=1}^n \tau_i^n(\theta^n) - c^n(\alpha^n(\theta^n))\Big|\Big) \to 0 \quad \text{as} \quad n \to \infty.$$

The following two propositions are the driving force behind Theorem 5, which shows that the cost-sharing pivotal mechanism is AEPBB. They are also useful results in their own right. The first says that the total CSP payment is bounded below by zero and above by $c(\alpha(\theta)) + c(K)$, and hence that the distance from EPBB in a CSP is bounded by $c(K)$. The second says that the probability that any individual is pivotal goes to zero as n goes to infinity.

Proposition 5. *Consider any environment $\mathcal{E}_F \in \mathbb{E}_F$. For any $\theta \in \Theta$, the total pivotal payment can be no less than zero and no more than $c(\alpha(\theta))$, and the total CSP payment can be no less than zero and no more than $c(\alpha(\theta)) + c(K)$. That is,*

1. $0 \leq \sum_{i=1}^n t_i(\theta) \leq c(\alpha(\theta))$ *and*
2. $0 \leq \sum_{i=1}^n \tau_i(\theta) \leq c(\alpha(\theta)) + c(K)$*,*

where $t_i(\theta)$ is i's transfer in a pivotal mechanism and $\tau_i(\theta) = t_i(\theta) + c_i(\alpha(0, \theta_{-i}))$ is i's transfer in a CSP mechanism. If the decision rule is monotone, the second upper bound can be lowered to $2c(\alpha(\theta))$.

Proof Sketch. With only two alternatives $\{0, 1\}$, the maximum revenue in a pivotal mechanism occurs when every individual is exactly pivotal, resulting in a revenue equal to the difference in cost between the two alternatives. Now, consider running a sequence of binary pivotal mechanisms on $\{0, \ldots, K\}$, where the selected alternative from $\{0, 1\}$ is run against 2, the selected alternative from that mechanism is run against 3, and so on.[42] Lemma 1 shows that the transfer in such a sequence of binary pivotal mechanisms is always larger than the transfer in a single pivotal mechanism on $\{0, \ldots, K\}$.[43] Hence the total revenue in a pivotal mechanism on $\{0, \ldots, K\}$ is less than the revenue from a sequence of binary pivotal mechanisms, each of which has a maximum revenue equal to the difference in cost between the two alternatives considered, so the pivotal mechanism raises no more than $c(\alpha(\theta))$ in revenue. In a CSP, the total fair-share payment $\sum_{i=1}^{n} c_i(\alpha(0, \theta_{-i}))$ can be no more than $c(K)$, and with a monotone decision rule, can be no more than $c(\alpha(\theta))$. ■

Proposition 6. *Given any sequence of finite environments $(\theta^n, z^n, \mathcal{E}^n)_{n\in\mathbb{N}}$ generated by i.i.d. draws from some $F \in \Delta(\Theta_0 \times Z_0)$, the probability that there is at least one pivotal player in $\mathcal{E}^n$ given θ^n goes to zero as n goes to infinity.*

Proof Commentary. The proof follows similar arguments as those found in Rob (1982), generalized for finite public good environments with arbitrary cost sequences. For the binary case, this involves showing that the probability that the maximum value θ_n^* in a sequence $\{\theta_i\}_{i=1}^{n}$ of non-negative i.i.d. random variables is greater than the distance between the summation $\sum_{i=1}^{n} \theta_i$ and *any* non-negative real number c_n, goes to zero. □

We may now show the cost-sharing pivotal mechanism is AEPBB.

Theorem 5. *Given any sequence of finite environments $(\theta^n, z^n, \mathcal{E}^n)_{n\in\mathbb{N}}$ generated by i.i.d. draws from some $F \in \Delta(\Theta_0 \times Z_0)$, any sequence of cost-sharing pivotal mechanisms $((\alpha^n, \tau^n))_{n\in\mathbb{N}}$ is asymptotically ex-post budget-balanced.*

Proof Sketch. Condition 1 of AEPBB follows by Proposition 6. Consider Condition 2. Suppose a binary public good environment for simplicity. The full proof considers any finite public good environment. The distance from EPBB in a CSP is bounded by $c(1)$ (Proposition 5). Suppose $\frac{c^n(1)}{n} \not\to \infty$ as $n \to \infty$. The CSP is EPBB if no individual is pivotal. The probability that any individual is pivotal goes to zero as n goes to infinity (Proposition 6). Suppose $\frac{c^n(1)}{n} \to \infty$ as $n \to \infty$. The CSP is EPBB if no good is produced.

[42] Viewed as a static mechanism, this mechanism is efficient and can be made to have the same decision rule as any pivotal mechanism on $\{0, \ldots, K\}$, though it is not strategy-proof.

[43] Roughly, you have more opportunities to accumulate pivotal payments in a sequence of binary pivotal mechanisms than in a single overall pivotal mechanism.

The probability that it is efficient to produce the good goes to zero faster than $\frac{c^n(1)}{n}$ goes to infinity by Chebyshev's inequality. ■

Given an efficient and monotone decision rule, there are many transfer rules for which the resulting mechanism satisfies AEPBB, strategy-proofness, and at least one of cost-sharing universal participation and the fair pricing principle. This is not surprising, as the asymptotic nature of AEPBB affords many mechanisms that behave similarly in the limit.

However, in a sense that the following theorem makes precise, all such transfer rules are simply perturbations from the CSP transfer rule whose deviations disappear as n goes to infinity. In particular, given any sequence of efficient and monotone decision rules, any AEPBB sequence of mechanisms satisfying strategy-proofness and at least one of cost-sharing universal participation and the fair pricing principle must have a transfer rule equal to that of the CSP plus a term h_i^n which is 1) non-negative, 2) zero with probability one in the limit, and 3) zero in expectation, averaged across individuals, in the limit. In this sense, the cost-sharing pivotal mechanism is effectively the unique mechanism which satisfies AEPBB, strategy-proofness, efficiency, and at least one of cost-sharing universal participation and the fair pricing principle.

Theorem 6. *Consider any sequence of finite environments $(\theta^n, z^n, \mathcal{E}^n)_{n\in\mathbb{N}}$ generated by i.i.d. draws from some $F \in \Delta(\Theta_0 \times Z_0)$. Let $(\alpha^n)_{n\in\mathbb{N}}$ be a sequence of efficient and monotone decision rules and $(\tau^n)_{n\in\mathbb{N}}$ be a sequence of CSP transfer rules. A sequence of efficient and monotone mechanisms $((\alpha^n, \hat{\tau}^n))_{n\in\mathbb{N}}$ satisfies AEPBB and strategy-proofness and cost-sharing universal participation for each n if and only if, for all i,*

$$\hat{\tau}_i^n(\theta^n) = \tau_i^n(\theta) + h_i^n(\theta_{-i}^n),$$

for some sequence of functions $h_i^n : \Theta_{-i} \to \mathbb{R}$ such that

1. *$h_i^n(\theta_{-i}^n) \leq 0$ for all $\theta_{-i}^n \in \Theta_{-i}^n$, i, and n,*
2. *$\lim_{n\to\infty} \mathbb{P}(h_i^n(\theta_{-i}^n) = 0$ for all $i) = 1$, and*
3. *$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \mathbb{E}\left[h_i^n(\theta_{-i}^n)\right] = 0$.*

The same holds replacing cost-sharing universal participation with the fair pricing principle per unit. Under Assumption 1, the same holds replacing the fair pricing principle per unit with the fair pricing principle.

Proof Sketch.

1. Follows from Holmström (1979), Theorem 1, and Theorem 3.
2. The probability that no agent is pivotal goes to one (Proposition 6), and the CSP is EPBB when no agent is pivotal. Hence, by (1), $\mathbb{P}(\text{EPBB}) \to 1 \iff \mathbb{P}(\sum_{i=1}^n h_i^n(\theta_{-i}^n) = 0 \mid \text{no one pivotal}) \to 1 \iff \mathbb{P}(h_i^n(\theta_{-i}^n) = 0 \text{ for all } i \mid \text{no one pivotal}) \to 1 \iff \mathbb{P}(h_i^n(\theta_{-i}^n) = 0 \text{ for all } i) \to 1$.

3. Follows from Theorem 5.

■

9 Conclusion

I have argued that the cost-sharing pivotal mechanism is a solution to the public good provision problem when the population is large and the government has the power to tax—and that it is effectively the unique solution. I conclude with an illustration. Consider what many would deem the simplest and most intuitive mechanism for providing public goods—select the efficient decision and have everyone pay their fair share of its cost. Let's call this a *share-the-cost mechanism (STC)*. The STC is simple. It is efficient, fair, and exactly budget-balanced.

But it has infinitely perverse incentives.[44] Consider the binary case. If your value for the good is greater than your fair share of its cost, then you want it to be produced—period. It is then weakly dominant to report an *infinitely large* value. If your value for the good is less than your fair share of its cost, then you want it not to be produced—period. It is then weakly dominant to report a *zero* value. Hence, the STC is far from being a viable solution to the public good provision problem.

Or is it? The cost-sharing pivotal mechanism is just a small tweak of the STC. The STC says that each individual should pay her fair share of *what is produced*. The CSP says that each individual should pay her fair share of what is produced, unless she is pivotal, in which case she should pay her fair share of *what would have been produced without her* plus enough to compensate the others for the welfare loss imposed on them. These only differ when an individual is pivotal. But the probability that *any* individual is pivotal goes to zero as the population gets large. It is striking that a small tweak which occurs with vanishing probability can shift incentives from infinitely perverse to fully strategy-proof—but that is precisely what the CSP does. The cost-sharing pivotal mechanism can thus be seen as a tweak of the share-the-cost mechanism which repairs its egregious manipulability while sacrificing little of its otherwise stellar properties.

References

Al-Najjar, Nabil I., and Rann Smorodinsky. 2000. "Provision of a public good with bounded cost." *Economics Letters*, 67(3): 297–301.

Clarke, Edward H. 1971. "Multipart Pricing of Public Goods." *Public Choice*, 11: 17–33.

[44]Hence, in practice it is none of efficient, fair, and exactly budget-balanced, since these properties all rely on truthful reporting.

Drexl, Moritz, and Andreas Kleiner. 2018. “Why Voting? A Welfare Analysis.” *American Economic Journal: Microeconomics*, 10(3): 253–271.

Fryxell, Loren K. 2023. “Budget-Balancing the Groves Mechanisms.” https://lorenfryxell.com/papers/budget-balancing-the-groves-mechanisms.pdf.

Gibbard, Allan. 1973. “Manipulation of Voting Schemes: A General Result.” *Econometrica*, 41(4): 587–601.

Green, Jerry R., and Jean-Jacques Laffont. 1979. *Incentives in Public Decision-Making.* Vol. 1, North-Holland Publishing Company.

Green, Jerry R., Elon Kohlberg, and Jean-Jacques Laffont. 1976. “Partial Equilibrium Approach to the Free-Rider Problem.” *Journal of Public Economics*, 6(4): 375–394.

Groves, Theodore. 1973. “Incentives in Teams.” *Econometrica*, 41(4): 617–631.

Hellwig, Martin F. 2003. “Public-Good Provision with Many Participants.” *Review of Economic Studies*, 70(3): 589–614.

Holmström, Bengt. 1979. “Groves’ Scheme on Restricted Domains.” *Econometrica*, 47(5): 1137–1144.

Kuzmics, Christoph, and Jan-Henrik Steg. 2017. “On public good provision mechanisms with dominant strategies and balanced budget.” *Journal of Economic Theory*, 170: 56–69.

Laffont, Jean-Jacques, and Eric S. Maskin. 1982. “The Theory of Incentives: An Overview.” In *Advances in Economic Theory.* Chapter 2, 31–94. Cambridge University Press.

Mailath, George J., and Andrew Postlewaite. 1990. “Asymmetric Information Bargaining Problems with Many Agents.” *Review of Economic Studies*, 57(3): 351–367.

Mas-Colell, Andreu, Michael D. Whinston, and Jerry R. Green. 1995. *Microeconomic Theory.* Oxford University Press.

Milgrom, Paul, and Chris Shannon. 1994. “Monotone Comparative Statics.” *Econometrica*, 62(1): 157–180.

Mitsui, Toshihide. 1983. “Asymptotic Efficiency of the Pivotal Mechanism with General Project Space.” *Journal of Economic Theory*, 31: 318–331.

Moulin, Hervé J. 1986. “Characterizations of the Pivotal Mechanism.” *Journal of Public Economics*, 53–78.

Moulin, Hervé J. 1988. *Axioms of Cooperative Decision Making.* Cambridge University Press.

Nath, Swaprava, and Tuomas Sandholm. 2019. "Efficiency and budget balance in general quasi-linear domains." *Games and Economic Behavior*, 113: 673–693.

Rob, Rafael. 1982. "Asymptotic Efficiency of the Demand Revealing Mechanism." *Journal of Economic Theory*, 28: 207–220.

Salanié, Bernard. 2017. *The Economics of Contracts: A Primer.* . 2nd ed., MIT Press.

Serizawa, Shigehiro. 1999. "Strategy-Proof and Symmetric Social Choice Functions for Public Good Economies." *Econometrica*, 67(1): 121–145.

van der Vaart, A. W. 1998. *Asymptotic Statistics.* Cambridge University Press.

Varian, Hal R. 1992. *Microeconomic Analysis.* . 3rd ed., Norton & Company, Inc.

Vickrey, William. 1961. "Counterspeculation, Auctions, and Competitive Sealed Tenders." *Journal of Finance*, 16(1): 8–37.

Xi, Jin, and Haitian Xie. 2021. "Strength in Numbers: Robust Mechanisms for Public Goods with Many Agents."

A Constructing Fair Cost Shares

In this appendix, I present some examples of how a government might construct fair shares from observable traits.

A simple notion of fairness is that of equal shares, which says that it is fair for everyone to pay an equal share of the cost: $\phi_i(y,z) = c(y)/n$ for all y and z. But as discussed in Section 4, fair shares may depend on anything which is observable to the government, including income level and distance from the public good.

Suppose income levels w_i are observable. An appealing notion of fairness might then be that it is fair for everyone to pay an equal share of their *income*:

$$\phi_i(y,z) = c(y)\frac{w_i}{\sum_j w_j}$$

for all y and z.

In fact, a natural generalization of this idea is that it is fair for everyone to pay an equal share of their *duty*:

$$\phi_i(y,z) = c(y)\frac{\delta(z_i)}{\sum_j \delta(z_j)},$$

where $Z_i = Z_0$ for all i and $\delta : Z_0 \to \mathbb{R}_+$ is an index measuring i's relative duty to pay for the public good.

If distance d_i from the public good is observable, an appealing index of relative duty might be one in which duty is constant within a particular radius r and is inversely proportional to distance beyond r,

$$\delta(d_i) = \begin{cases} 1/r & \text{if } 0 \leq d_i \leq r \\ 1/d_i & \text{if } d_i > r \end{cases} .$$

If both income w_i and distance d_i are observable, an appealing index of relative duty might be

$$\delta(w_i, d_i) = \begin{cases} w_i/r & \text{if } 0 \leq d_i \leq r \\ w_i/d_i & \text{if } d_i > r \end{cases} ,$$

so that doubling one's income doubles one's duty and doubling one's distance halves one's duty beyond radius r.

B Proofs Omitted from Main Body

Theorem 3. *Consider any environment $\mathcal{E}_C \in \mathbb{E}_C$ and any efficient and monotone decision rule α. A mechanism $f = (\alpha, \tau)$ maximizes ex-post revenue among all mechanisms which satisfy strategy-proofness, efficiency, and the fair pricing principle per unit if and only if it is a cost-sharing pivotal mechanism. Under Assumption 1, the same holds replacing the fair pricing principle per unit with the fair pricing principle.*

Proof. Consider any environment $\mathcal{E}_C \in \mathbb{E}_C$. By Holmström (1979), a mechanism f is strategy-proof and efficient if and only if it is a Groves mechanism. Let $f = (\alpha, \tau)$ be a Groves mechanism with a monotone decision rule α and $\tau_i(\theta) = g_i(\theta_{-i}) - [(\sum_{j\neq i} v_j(\alpha(\theta), \theta_j)) - c(\alpha(\theta))]$ for some $g_i : \Theta_{-i} \to \mathbb{R}$.

Part I. We would like to show that $g_i(\theta_{-i}) = [(\sum_{j\neq i} v_j(\alpha(0, \theta_{-i}), \theta_j)) - c(\alpha(0, \theta_{-i}))] + c_i(\alpha(0, \theta_{-i}))$ maximizes ex post revenue subject to the FPP per unit. The FPP per unit requires that for all i and θ, $\int_0^{\alpha(\theta)} \max\{v_i'(y, \theta_i),\, c_i'(y)\}\, \mathrm{d}y + [(\sum_{j\neq i} v_j(\alpha(\theta), \theta_j)) - c(\alpha(\theta))] \geq g_i(\theta_{-i})$, so to maximize ex-post revenue subject to the FPP per unit, set

$$g_i(\theta_{-i}) = \inf_{\theta_i \in \Theta_i} \Big\{ \int_0^{\alpha(\theta)} \max\left\{v_i'(y, \theta_i),\, c_i'(y)\right\} + \sum_{j\neq i} v_j'(y, \theta_j) - c'(y)\ \mathrm{d}y \Big\}.$$

We would like to show that for any θ, increasing $v_i'(y, \theta_i)$ pointwise weakly increases the objective function, and hence that $0 \in \Theta_i$ is a minimizer. Suppose $v_i'(y, \hat{\theta}_i) \geq v_i'(y, \theta_i)$ for all $y \geq 0$. Since α is monotonic, $\alpha(\hat{\theta}_i, \theta_{-i}) \geq \alpha(\theta)$. We would like to show $\int_0^{\alpha(\hat{\theta}_i, \theta_{-i})} \max\{v_i'(y, \hat{\theta}_i),\, c_i'(y)\}+$

$\sum_{j\neq i} v_j'(y,\theta_j) - c'(y)\ \mathrm{d}y \geq \int_0^{\alpha(\theta)} \max\{v_i'(y,\theta_i),\, c_i'(y)\} + \sum_{j\neq i} v_j'(y,\theta_j) - c'(y)\ \mathrm{d}y$, or equivalently,

$$\int_0^{\alpha(\theta)} \max\left\{v_i'(y,\hat{\theta}_i),\, c_i'(y)\right\} - \max\left\{v_i'(y,\theta_i),\, c_i'(y)\right\}\ \mathrm{d}y$$
$$+ \int_{\alpha(\theta)}^{\alpha(\hat{\theta}_i,\theta_{-i})} \max\left\{v_i'(y,\hat{\theta}_i),\, c_i'(y)\right\} + \sum_{j\neq i} v_j'(y,\theta_j) - c'(y)\ \mathrm{d}y \geq 0.$$

The first term is non-negative, and by definition of α,

$$\begin{aligned}
\int_0^{\alpha(\hat{\theta}_i,\theta_{-i})} v_i'(y,\hat{\theta}_i) + \sum_{j\neq i} v_j'(y,\theta_j) - c'(y)\ \mathrm{d}y &\geq \int_0^{\alpha(\theta)} v_i'(y,\hat{\theta}_i) + \sum_{j\neq i} v_j'(y,\theta_j) - c'(y)\ \mathrm{d}y \\
\iff \int_{\alpha(\theta)}^{\alpha(\hat{\theta}_i,\theta_{-i})} v_i'(y,\hat{\theta}_i) + \sum_{j\neq i} v_j'(y,\theta_j) - c'(y)\ \mathrm{d}y &\geq 0 \\
\implies \int_{\alpha(\theta)}^{\alpha(\hat{\theta}_i,\theta_{-i})} \max\left\{v_i'(y,\hat{\theta}_i),\, c_i'(y)\right\} + \sum_{j\neq i} v_j'(y,\theta_j) - c'(y)\ \mathrm{d}y &\geq 0.
\end{aligned}$$

Hence, $0 \in \Theta_i$ is a minimizer and for all θ_{-i},

$$\begin{aligned}
g_i(\theta_{-i}) &= \int_0^{\alpha(0,\theta_{-i})} \max\left\{v_i'(y,0),\, c_i'(y)\right\} + \sum_{j\neq i} v_j'(y,\theta_j) - c'(y)\ \mathrm{d}y \\
&= \left[\Big(\sum_{j\neq i} v_j(\alpha(0,\theta_{-i}),\theta_j)\Big) - c(\alpha(0,\theta_{-i}))\right] + c_i(\alpha(0,\theta_{-i})).
\end{aligned}$$

Part II. Suppose Assumption 1. We would like to show that $g_i(\theta_{-i}) = [(\sum_{j\neq i} v_j(\alpha(0,\theta_{-i}),\theta_j)) - c(\alpha(0,\theta_{-i}))] + c_i(\alpha(0,\theta_{-i}))$ maximizes ex-post revenue subject to the FPP. The FPP requires that for all i and θ, $\max\{v_i(\alpha(\theta),\theta_i),\, c_i(\alpha(\theta))\} + [(\sum_{j\neq i} v_j(\alpha(\theta),\theta_j)) - c(\alpha(\theta))] \geq g_i(\theta_{-i})$, so to maximize ex-post revenue subject to the FPP, set

$$g_i(\theta_{-i}) = \inf_{\theta_i\in\Theta_i} \left\{ \max\left\{ \int_0^{\alpha(\theta)} v_i'(y,\theta_i)\ \mathrm{d}y,\, \int_0^{\alpha(\theta)} c_i'(y)\ \mathrm{d}y \right\} + \int_0^{\alpha(\theta)} \sum_{j\neq i} v_j'(y,\theta_j) - c'(y)\ \mathrm{d}y \right\}.$$

We would like to show that for any θ, increasing $v_i'(y,\theta_i)$ pointwise weakly increases the objective function, and hence that $0 \in \Theta_i$ is a minimizer. Suppose $v_i'(y,\hat{\theta}_i) \geq v_i'(y,\theta_i)$ for all $y \geq 0$. Since α is monotonic, $\alpha(\hat{\theta}_i,\theta_{-i}) \geq \alpha(\theta)$. We would like to show

$$\max\left\{ \int_0^{\alpha(\hat{\theta}_i,\theta_{-i})} v_i'(y,\hat{\theta}_i)\ \mathrm{d}y,\, \int_0^{\alpha(\hat{\theta}_i,\theta_{-i})} c_i'(y)\ \mathrm{d}y \right\} + \int_0^{\alpha(\hat{\theta}_i,\theta_{-i})} \sum_{j\neq i} v_j'(y,\theta_j) - c'(y)\ \mathrm{d}y$$
$$\geq \max\left\{ \int_0^{\alpha(\theta)} v_i'(y,\theta_i)\ \mathrm{d}y,\, \int_0^{\alpha(\theta)} c_i'(y)\ \mathrm{d}y \right\} + \int_0^{\alpha(\theta)} \sum_{j\neq i} v_j'(y,\theta_j) - c'(y)\ \mathrm{d}y$$

which is equivalent to

$$\max\Big\{\int_0^{\alpha(\hat\theta_i,\theta_{-i})} v_i'(y,\hat\theta_i)\,\mathrm{d}y,\ \int_0^{\alpha(\hat\theta_i,\theta_{-i})} c_i'(y)\,\mathrm{d}y\Big\} - \max\Big\{\int_0^{\alpha(\theta)} v_i'(y,\theta_i)\,\mathrm{d}y,\ \int_0^{\alpha(\theta)} c_i'(y)\,\mathrm{d}y\Big\} + \int_{\alpha(\theta)}^{\alpha(\hat\theta_i,\theta_{-i})} \sum_{j\neq i} v_j'(y,\theta_j) - c'(y)\ \mathrm{d}y \geq 0.$$

Notice that, since $\int_0^{\alpha(\hat\theta_i,\theta_{-i})} v_i'(y,\hat\theta_i)+\sum_{j\neq i} v_j'(y,\theta_j)-c'(y)\,\mathrm{d}y \geq \int_0^{\alpha(\theta)} v_i'(y,\hat\theta_i)+\sum_{j\neq i} v_j'(y,\theta_j)-c'(y)\ \mathrm{d}y$,

$$\int_{\alpha(\theta)}^{\alpha(\hat\theta_i,\theta_{-i})} v_i'(y,\hat\theta_i) + \sum_{j\neq i} v_j'(y,\theta_j) - c'(y)\ \mathrm{d}y \geq 0.$$

Hence, it is sufficient to show

$$\max\Big\{\int_0^{\alpha(\hat\theta_i,\theta_{-i})} v_i'(y,\hat\theta_i)\,\mathrm{d}y,\ \int_0^{\alpha(\hat\theta_i,\theta_{-i})} c_i'(y)\,\mathrm{d}y\Big\} - \max\Big\{\int_0^{\alpha(\theta)} v_i'(y,\theta_i)\,\mathrm{d}y,\ \int_0^{\alpha(\theta)} c_i'(y)\,\mathrm{d}y\Big\} \geq \int_{\alpha(\theta)}^{\alpha(\hat\theta_i,\theta_{-i})} v_i'(y,\hat\theta_i)\ \mathrm{d}y. \quad (2)$$

Case 1. Suppose $\int_0^{\alpha(\hat\theta_i,\theta_{-i})} v_i'(y,\hat\theta_i)\,\mathrm{d}y < \int_0^{\alpha(\hat\theta_i,\theta_{-i})} c_i'(y)\,\mathrm{d}y$ and $\int_0^{\alpha(\theta)} v_i'(y,\theta_i)\,\mathrm{d}y < \int_0^{\alpha(\theta)} c_i'(y)\,\mathrm{d}y$. Then (2) becomes $\int_0^{\alpha(\hat\theta_i,\theta_{-i})} c_i'(y)\ \mathrm{d}y - \int_0^{\alpha(\theta)} c_i'(y)\ \mathrm{d}y \geq \int_{\alpha(\theta)}^{\alpha(\hat\theta_i,\theta_{-i})} v_i'(y,\hat\theta_i)\ \mathrm{d}y$, which holds if and only if $\int_{\alpha(\theta)}^{\alpha(\hat\theta_i,\theta_{-i})} c_i'(y)\ \mathrm{d}y \geq \int_{\alpha(\theta)}^{\alpha(\hat\theta_i,\theta_{-i})} v_i'(y,\hat\theta_i)\ \mathrm{d}y$, which is implied by the first inequality and Assumption 1.

Case 2. Suppose $\int_0^{\alpha(\hat\theta_i,\theta_{-i})} v_i'(y,\hat\theta_i)\,\mathrm{d}y < \int_0^{\alpha(\hat\theta_i,\theta_{-i})} c_i'(y)\,\mathrm{d}y$ and $\int_0^{\alpha(\theta)} v_i'(y,\theta_i)\,\mathrm{d}y \geq \int_0^{\alpha(\theta)} c_i'(y)\,\mathrm{d}y$. Then (2) becomes $\int_0^{\alpha(\hat\theta_i,\theta_{-i})} c_i'(y)\ \mathrm{d}y - \int_0^{\alpha(\theta)} v_i'(y,\theta_i)\ \mathrm{d}y \geq \int_{\alpha(\theta)}^{\alpha(\hat\theta_i,\theta_{-i})} v_i'(y,\hat\theta_i)\ \mathrm{d}y$ which is implied by the first inequality and the definition of $\hat\theta_i$.

Case 3. Suppose $\int_0^{\alpha(\hat\theta_i,\theta_{-i})} v_i'(y,\hat\theta_i)\,\mathrm{d}y \geq \int_0^{\alpha(\hat\theta_i,\theta_{-i})} c_i'(y)\,\mathrm{d}y$ and $\int_0^{\alpha(\theta)} v_i'(y,\theta_i)\,\mathrm{d}y < \int_0^{\alpha(\theta)} c_i'(y)\,\mathrm{d}y$. Then (2) becomes $\int_0^{\alpha(\hat\theta_i,\theta_{-i})} v_i'(y,\hat\theta_i)\ \mathrm{d}y - \int_0^{\alpha(\theta)} c_i'(y)\ \mathrm{d}y \geq \int_{\alpha(\theta)}^{\alpha(\hat\theta_i,\theta_{-i})} v_i'(y,\hat\theta_i)\ \mathrm{d}y$, which holds if and only if $\int_0^{\alpha(\theta)} v_i'(y,\hat\theta_i)\ \mathrm{d}y \geq \int_0^{\alpha(\theta)} c_i'(y)\ \mathrm{d}y$, which is implied by the first inequality and Assumption 1.

Case 4. Suppose $\int_0^{\alpha(\hat\theta_i,\theta_{-i})} v_i'(y,\hat\theta_i)\,\mathrm{d}y \geq \int_0^{\alpha(\hat\theta_i,\theta_{-i})} c_i'(y)\,\mathrm{d}y$ and $\int_0^{\alpha(\theta)} v_i'(y,\theta_i)\,\mathrm{d}y \geq \int_0^{\alpha(\theta)} c_i'(y)\,\mathrm{d}y$. Then (2) becomes $\int_0^{\alpha(\hat\theta_i,\theta_{-i})} v_i'(y,\hat\theta_i)\ \mathrm{d}y - \int_0^{\alpha(\theta)} v_i'(y,\theta_i)\ \mathrm{d}y \geq \int_{\alpha(\theta)}^{\alpha(\hat\theta_i,\theta_{-i})} v_i'(y,\hat\theta_i)\ \mathrm{d}y$, which holds by the definition of $\hat\theta_i$.

Hence, $0 \in \Theta_i$ is a minimizer and for all θ_{-i},

$$\begin{aligned} g_i(\theta_{-i}) &= \max\Big\{\int_0^{\alpha(0,\theta_{-i})} v_i'(y,0)\ \mathrm{d}y,\ \int_0^{\alpha(0,\theta_{-i})} c_i'(y)\ \mathrm{d}y\Big\} + \int_0^{\alpha(0,\theta_{-i})} \sum_{j\neq i} v_j'(y,\theta_j) - c'(y)\ \mathrm{d}y \\ &= \Big[\Big(\sum_{j\neq i} v_j(\alpha(0,\theta_{-i}),\theta_j)\Big) - c(\alpha(0,\theta_{-i}))\Big] + c_i(\alpha(0,\theta_{-i})). \end{aligned}$$

■

Theorem 4. *Consider any environment $\mathcal{E}_C \in \mathbb{E}_C$. Suppose that for any i and θ_{-i}, $\max A^*(0, \theta_{-i})$ exists and there exists $\theta_i \in \Theta_i$ such that*

$$v_i'(y, \theta_i) = \begin{cases} \phi_i(y) & \text{if } y \leq \max A^*(0, \theta_{-i}) \\ 0 & \text{if } y > \max A^*(0, \theta_{-i}) \end{cases}, \tag{1}$$

where $\phi_i(y) \geq \max_{j \neq i} v_j'(y, \theta_j)$ and $\phi_i(y) > 0$ for all y.[45] *For any efficient and monotone decision rule α, there is no strategy-proof and cost-sharing universal participation mechanism that is no farther from ex-post budget-balance than a cost-sharing pivotal mechanism for every θ. The same holds replacing cost-sharing universal participation with the fair pricing principle per unit. Under Assumption 1, the same holds replacing the fair pricing principle per unit with the fair pricing principle.*

Proof. Let

$$\begin{aligned} t_i(\theta) &= \left[\Big(\sum_{j \neq i} v_j(\alpha(0, \theta_{-i}), \theta_j)\Big) - c(\alpha(0, \theta_{-i}))\right] - \left[\Big(\sum_{j \neq i} v_j(\alpha(\theta), \theta_j)\Big) - c(\alpha(\theta))\right] \\ &= \int_0^{\alpha(0,\theta_{-i})} \sum_{j \neq i} v_j'(y, \theta_j) - c'(y) \, \mathrm{d}y - \int_0^{\alpha(\theta)} \sum_{j \neq i} v_j'(y, \theta_j) - c'(y) \, \mathrm{d}y \end{aligned}$$

be i's pivotal payment and $\tau_i(\theta) = t_i(\theta) + c_i(\alpha(0, \theta_{-i}))$ be i's CSP transfer.

Part I. First, we would like to show that for any θ_{-i}, there exists θ_i such that a CSP mechanism does not run a strict budget surplus—i.e., $\sum_{i=1}^n \tau_i(\theta) \leq c(\alpha(\theta))$.

For any θ, $y^* \in A^*(\theta)$ if and only if for any $y' \leq y^* \leq y''$,

$$\int_{y'}^{y^*} \sum_{i \in I} v_i'(y, \theta_i) - c'(y) \, \mathrm{d}y \geq 0 \geq \int_{y^*}^{y''} \sum_{i \in I} v_i'(y, \theta_i) - c'(y) \, \mathrm{d}y.$$

That is, y^* is efficient if and only if moving from any $y' \leq y^*$ to y^* weakly increases welfare and moving from y^* to any $y'' \geq y^*$ weakly decreases welfare.

Fix any θ_{-i}. Let θ_i satisfy (1). Then for any $y' < \max A^*(0, \theta_{-i}) < y''$,

$$\int_{y'}^{\max A^*(0,\theta_{-i})} \sum_{k \neq i} v_k'(y, \theta_k) - c'(y) \, \mathrm{d}y \geq 0 > \int_{\max A^*(0,\theta_{-i})}^{y''} \sum_{k \neq i} v_k'(y, \theta_i) - c'(y) \, \mathrm{d}y \tag{3}$$

where the strict inequality follows since we are considering $\max A^*(0, \theta_{-i})$. By (3), for any $y' < \max A^*(0, \theta_{-i}) < y''$,

$$\int_{y'}^{\max A^*(0,\theta_{-i})} \sum_{k=1}^{n} v_k'(y, \theta_k) - c'(y) \, \mathrm{d}y > 0 > \int_{\max A^*(0,\theta_{-i})}^{y''} \sum_{k=1}^{n} v_k'(y, \theta_i) - c'(y) \, \mathrm{d}y$$

[45] This is a weak richness condition on the domain of values which says that each individual may value marginal units of the public good as much as any other individual and that this marginal value may drop to zero at any point.

where the first strict inequality follows since $\phi(y) > 0$ for all $y \leq A^*(0, \theta_{-i})$. Hence, $\alpha(\theta) = \max A^*(0, \theta_{-i})$. Since $\alpha(0, \theta_{-i}), \alpha(\theta) \in A^*(0, \theta_{-i})$,

$$\int_0^{\alpha(0,\theta_{-i})} \sum_{k \neq i} v_k'(y, \theta_k) - c'(y) \text{ d}y = \int_0^{\alpha(\theta)} \sum_{k \neq i} v_k'(y, \theta_k) - c'(y) \text{ d}y,$$

so i's pivotal payment is zero and $\tau_i(\theta) = c_i(\alpha(0, \theta_{-i})) \leq c_i(\alpha(\theta))$ by monotonicity. By (3), for any $y' < \max A^*(0, \theta_{-i}) < y''$,

$$\int_{y'}^{\max A^*(0,\theta_{-i})} \sum_{k \neq j} v_k'(y, \theta_k) - c'(y) \text{ d}y \geq 0 > \int_{\max A^*(0,\theta_{-i})}^{y''} \sum_{k \neq j} v_k'(y, \theta_i) - c'(y) \text{ d}y$$

by definition of $v_i'(y, \theta_i)$. Hence, $\alpha(\theta) = \max A^*(0, \theta_{-i}) \in A^*(0, \theta_{-j})$. Since $\alpha(0, \theta_{-j}), \alpha(\theta) \in A^*(0, \theta_{-j})$,

$$\int_0^{\alpha(0,\theta_{-j})} \sum_{k \neq j} v_k'(y, \theta_k) - c'(y) \text{ d}y = \int_0^{\alpha(\theta)} \sum_{k \neq j} v_k'(y, \theta_k) - c'(y) \text{ d}y,$$

so j's pivotal payment is zero and $\tau_j(\theta) = c_j(\alpha(0, \theta_{-j})) \leq c_j(\alpha(\theta))$ by monotonicity. Hence, $\sum_{i=1}^n \tau_i(\theta) \leq c(\alpha(\theta))$ as desired.

Part II. Consider any environment $\mathcal{E}_C \in \mathbb{E}_C$. By Holmström (1979), a mechanism f is strategy-proof and efficient if and only if it is a Groves mechanism. We may write i's Groves transfer as the sum of her pivotal payment and a term that depends only on her opponents' reports, $t_i(\theta) + h_i(\theta_{-i})$ for some $h_i : \Theta_{-i} \to \mathbb{R}$. For any i and θ_{-i}, if θ_i satisfies (1), then setting $h_i(\theta_{-i}) < c_i(\alpha(0, \theta_{-i}))$ runs a strictly larger budget-deficit than a CSP by Part I, and setting $h_i(\theta_{-i}) > c_i(\alpha(0, \theta_{-i}))$ violates CS-UP by Theorem 1 and the FPP per unit (the FPP under Assumption 1) by Theorem 3. ■

We now proceed to prove Proposition 5, Proposition 6, and Theorem 5.

Consider any environment $\mathcal{E}_F \in \mathbb{E}_F$ and any efficient decision rule $\alpha : \Theta \to Y$. For any $Z \subseteq Y = \{0, 1, \ldots, K\}$, let $A^*(\theta; Z) = \arg\max_{y \in Z} \sum_{i \in I} v_i(y, \theta_i) - c(y)$,

$$\hat{\alpha}(\theta; Z) = \begin{cases} \alpha(\theta) & \text{if } \alpha(\theta) \in Z \\ \max A^*(\theta; Z) & \text{otherwise} \end{cases},$$

and

$$t_i(\theta; Z) = \Big[\Big(\sum_{j \neq i} v_j(\hat{\alpha}(0, \theta_{-i}; Z), \theta_j)\Big) - c(\hat{\alpha}(0, \theta_{-i}; Z))\Big] - \Big[\Big(\sum_{j \neq i} v_j(\hat{\alpha}(\theta; Z), \theta_j)\Big) - c(\hat{\alpha}(\theta; Z))\Big].$$

$A^*(\theta; Z)$ is the set of efficient alternatives within Z. $\hat{\alpha}(\theta; Z)$ is the decision rule that selects according to α when $\alpha(\theta)$ is in Z and selects the largest efficient alternative in Z otherwise. $t_i(\theta; Z)$ is the pivot transfer rule associated with $\hat{\alpha}(\theta; Z)$. For any $k \le K$, let

$$
\begin{aligned}
t_i^S(\theta; \{0, 1, \ldots, k\}) = t_i(\theta; \{0, 1\}) &+ t_i(\theta; \{\hat{\alpha}(\theta; \{0, 1\}), 2\}) \\
&+ t_i(\theta; \{\hat{\alpha}(\theta; \{\hat{\alpha}(\theta; \{0, 1\}), 2\}), 3\}) + \ldots + t_i(\theta; \{\hat{\alpha}(\theta; \ldots), k\})
\end{aligned}
$$

be the total transfer associated with running a sequence of binary pivotal mechanisms, where the selected alternative from $\{0, 1\}$ is run against 2, the selected alternative from that mechanism is run against 3, and so on. Let $t_i(\theta) = t_i(\theta; Y)$ and $t_i^S(\theta) = t_i^S(\theta; Y)$ be the pivot transfer and the sequential pivot transfer for i, respectively.

Lemma 1. *Given any environment $\mathcal{E}_F \in \mathbb{E}_F$, $t_i^S(\theta) \ge t_i(\theta)$ for all θ.*

Proof. First notice that

$$
\begin{aligned}
t_i^S(\theta) = t_i(\theta; \{0, 1\}) &+ t_i(\theta; \{\hat{\alpha}(\theta; \{0, 1\}), 2\}) \\
&+ t_i(\theta; \{\hat{\alpha}(\theta; \{0, 1, 2\}), 3\}) + \ldots + t_i(\theta; \{\hat{\alpha}(\theta; \{0, 1, \ldots, A-1\}), A\}).
\end{aligned}
$$

We now proceed by induction.

Step 1. We would like to show $t_i^S(\theta; \{0, 1\}) \ge t_i(\theta; \{0, 1\})$. By definition, $t_i^S(\theta; \{0, 1\}) = t_i(\theta; \{0, 1\})$.

Step 2. Suppose by induction that $t_i^S(\theta; \{0, \ldots, a\}) \ge t_i(\theta; \{0, \ldots, a\})$. We would like to show that $t_i^S(\theta; \{0, \ldots, a+1\}) \ge t_i(\theta; \{0, \ldots, a+1\})$.

Case 1. Suppose $\hat{\alpha}(0, \theta_{-i}; \{0, \ldots, a+1\}) = a+1$. Then $y^* = \hat{\alpha}(\theta; \{0, \ldots, a+1\}) \le a+1$, and by Proposition 3, $y^* \in A^*(0, \theta_{-i}; \{0, \ldots, a+1\})$. Hence, $t_i(\theta; \{0, \ldots, a+1\}) = 0$. The result follows since pivotal payments are non-negative.

Case 2. Suppose $\hat{\alpha}(0, \theta_{-i}; \{0, \ldots, a+1\}) < a+1$. We have

$$
\begin{aligned}
t_i(\theta; \{0, \ldots, a+1\}) = &\Big[\Big(\sum_{j \ne i} v_j(\hat{\alpha}(0, \theta_{-i}; \{0, \ldots, a+1\}), \theta_j)\Big) - c(\hat{\alpha}(0, \theta_{-i}; \{0, \ldots, a+1\}))\Big] \\
&- \Big[\Big(\sum_{j \ne i} v_j(\hat{\alpha}(\theta; \{0, \ldots, a+1\}), \theta_j)\Big) - c(\hat{\alpha}(\theta; \{0, \ldots, a+1\}))\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
&t_i^S(\theta; \{0, \ldots, a+1\}) = t_i^S(\theta; \{0, \ldots, a\}) \\
&\quad + \Big[\Big(\sum_{j \ne i} v_j(\hat{\alpha}(0, \theta_{-i}; \hat{\alpha}(\theta; \{0, \ldots, a\}) \cup \{a+1\}), \theta_j)\Big) - c(\hat{\alpha}(0, \theta_{-i}; \hat{\alpha}(\theta; \{0, \ldots, a\}) \cup \{a+1\}))\Big] \\
&\quad - \Big[\Big(\sum_{j \ne i} v_j(\hat{\alpha}(\theta; \hat{\alpha}(\theta; \{0, \ldots, a\}) \cup \{a+1\}), \theta_j)\Big) - c(\hat{\alpha}(\theta; \hat{\alpha}(\theta; \{0, \ldots, a\}) \cup \{a+1\}))\Big].
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&t_i^S(\theta;\{0,\dots,a+1\}) - t_i(\theta;\{0,\dots,a+1\}) = t_i^S(\theta;\{0,\dots,a\}) \\
&\quad + \Big[\Big(\sum_{j\neq i} v_j(\hat\alpha(0,\theta_{-i};\hat\alpha(\theta;\{0,\dots,a\})\cup\{a+1\}),\theta_j)\Big) - c(\hat\alpha(0,\theta_{-i};\hat\alpha(\theta;\{0,\dots,a\})\cup\{a+1\}))\Big] \\
&\quad - \Big[\Big(\sum_{j\neq i} v_j(\hat\alpha(0,\theta_{-i};\{0,\dots,a+1\}),\theta_j)\Big) - c(\hat\alpha(0,\theta_{-i};\{0,\dots,a+1\}))\Big] \\
&\geq \Big[\Big(\sum_{j\neq i} v_j(\hat\alpha(0,\theta_{-i};\{0,\dots,a\}),\theta_j)\Big) - c(\hat\alpha(0,\theta_{-i};\{0,\dots,a\}))\Big] \\
&\qquad - \Big[\Big(\sum_{j\neq i} v_j(\hat\alpha(\theta;\{0,\dots,a\}),\theta_j)\Big) - c(\hat\alpha(\theta;\{0,\dots,a\}))\Big] \\
&\quad + \Big[\Big(\sum_{j\neq i} v_j(\hat\alpha(0,\theta_{-i};\hat\alpha(\theta;\{0,\dots,a\})\cup\{a+1\}),\theta_j)\Big) - c(\hat\alpha(0,\theta_{-i};\hat\alpha(\theta;\{0,\dots,a\})\cup\{a+1\}))\Big] \\
&\quad - \Big[\Big(\sum_{j\neq i} v_j(\hat\alpha(0,\theta_{-i};\{0,\dots,a+1\}),\theta_j)\Big) - c(\hat\alpha(0,\theta_{-i};\{0,\dots,a+1\}))\Big] \\
&= -\Big[\Big(\sum_{j\neq i} v_j(\hat\alpha(\theta;\{0,\dots,a\}),\theta_j)\Big) - c(\hat\alpha(\theta;\{0,\dots,a\}))\Big] \\
&\quad + \Big[\Big(\sum_{j\neq i} v_j(\hat\alpha(0,\theta_{-i};\hat\alpha(\theta;\{0,\dots,a\})\cup\{a+1\}),\theta_j)\Big) - c(\hat\alpha(0,\theta_{-i};\hat\alpha(\theta;\{0,\dots,a\})\cup\{a+1\}))\Big] \\
&\geq 0,
\end{aligned}
$$

where the first equality follows by $\hat\alpha(\theta;\{0,\dots,a+1\}) = \hat\alpha(\theta;\hat\alpha(\theta;\{0,\dots,a\})\cup\{a+1\})$, the first inequality by the inductive hypothesis, and the final equality and inequality by the Case 2 assumption. ■

Proposition 5. *Consider any environment $\mathcal{E}_F \in \mathbb{E}_F$. For any $\theta \in \Theta$, the total pivotal payment can be no less than zero and no more than $c(\alpha(\theta))$, and the total CSP payment can be no less than zero and no more than $c(\alpha(\theta)) + c(K)$. That is,*

1. $0 \leq \sum_{i=1}^n t_i(\theta) \leq c(\alpha(\theta))$ *and*
2. $0 \leq \sum_{i=1}^n \tau_i(\theta) \leq c(\alpha(\theta)) + c(K)$,

where $t_i(\theta)$ is i's transfer in a pivotal mechanism and $\tau_i(\theta) = t_i(\theta) + c_i(\alpha(0,\theta_{-i}))$ is i's transfer in a CSP mechanism. If the decision rule is monotone, the second upper bound can be lowered to $2c(\alpha(\theta))$.

Proof. **Part I.** *Step 1.* First, we show $0 \leq \sum_{i=1}^n t_i(\theta) \leq c(\alpha(\theta))$ for any binary environment $\mathcal{E}_B \in \mathbb{E}_B$. Consider any $\theta, \hat\theta \in \Theta$ where $v_k(1,\hat\theta_k) < v_k(1,\theta_k)$ for some k and $v_j(1,\hat\theta_j) = v_j(1,\theta_j)$ for all $j \neq k$.

1. If $\sum_{i=1}^n v_i(1,\theta_i) < c(1)$, $\sum_{i=1}^n t_i(\theta) = \sum_{i=1}^n t_i(\hat\theta) = 0$.

2. If $\sum_{i=1}^n v_i(1,\theta_i) \geq c(1)$ and $\sum_{i=1}^n v_i(1,\hat{\theta}_i) < c(1)$, $\sum_{i=1}^n t_i(\theta) \geq 0 = \sum_{i=1}^n t_i(\hat{\theta})$.

3. If $\sum_{i=1}^n v_i(1,\theta_i) \geq c(1)$ and $\sum_{i=1}^n v_i(1,\hat{\theta}_i) \geq c(1)$, $\sum_{i=1}^n t_i(\hat{\theta}) \geq \sum_{i=1}^n t_i(\theta) \geq 0$.

To see (3), notice that k's pivotal payment remains unchanged between θ and $\hat{\theta}$. Consider any $j \neq k$. If j is pivotal under θ_{-j}, then she remains pivotal under $\hat{\theta}_{-j}$ and her pivotal payment strictly increases. If j is not pivotal under θ_{-j} and remains not pivotal under $\hat{\theta}_{-j}$, her pivotal payment remains zero. If j is not pivotal under θ_{-j}, but is pivotal under $\hat{\theta}_{-j}$, her payment increases from zero to some positive amount.

Hence, to maximize $\sum_{i=1}^n t_i(\theta)$ with respect to θ, it must be that $\sum_{i=1}^n v_i(1,\theta_i) = c(1)$ and the good is produced, i.e., every i is exactly pivotal. In this case, each i's pivotal payment is $v_i(1,\theta_i)$, and the total pivotal payment is $c(1)$. Since the total pivotal payment is zero when the good is not produced, the result follows.

Step 2. We now show this holds for any finite environment $\mathcal{E}_F \in \mathbb{E}_F$. To show that the total pivotal payment is non-negative, note that individual pivotal payments are themselves non-negative. To show that the total pivotal payment is no more than $c(K)$, note that

$$
\begin{aligned}
\sum_{i=1}^n t_i(\theta) &\leq \sum_{i=1}^n t_i^S(\theta) \\
&= \sum_{i=1}^n t_i(\theta;\{0,1\}) + t_i(\theta;\{\hat{\alpha}(\theta;\{0,1\}),2\}) \\
&\qquad + t_i(\theta;\{\hat{\alpha}(\theta;\{\hat{\alpha}(\theta;\{0,1\}),2\}),3\}) + \ldots + t_i(\theta;\{\hat{\alpha}(\theta;\ldots),K\}) \\
&\leq c(\hat{\alpha}(\theta;\{0,1\})) - c(0) \\
&\qquad + c(\hat{\alpha}(\theta;\{\hat{\alpha}(\theta;\{0,1\}),2\})) - c(\hat{\alpha}(\theta;\{0,1\})) \\
&\qquad + c(\hat{\alpha}(\theta;\{\hat{\alpha}(\theta;\{\hat{\alpha}(\theta;\{0,1\}),2\}),3\})) - c(\hat{\alpha}(\theta;\{\hat{\alpha}(\theta;\{0,1\}),2\})) \\
&\qquad + \ldots + c(\hat{\alpha}(\theta;\{\hat{\alpha}(\theta;\ldots),K\})) - c(\hat{\alpha}(\theta;\{\hat{\alpha}(\theta;\ldots),K-1\})) \\
&= c(\hat{\alpha}(\theta;\{\hat{\alpha}(\theta;\ldots),K\})) - c(0) \\
&= c(\alpha(\theta)),
\end{aligned}
$$

where the first inequality follows by Lemma 1 and the second inequality follows by Step 1.

Part II. Pivotal payments and fair-share payments are always non-negative. Hence, the total CSP payment is no less than zero.

The total pivotal payment $\sum_{i=1}^n t_i(\theta)$ is no more than $c(\alpha(\theta))$ by Part I. The total fair share payment $\sum_{i=1}^n c_i(\alpha(0,\theta_{-i}))$ can be no more than $c(K)$, since $\sum_{i=1}^n c_i(\alpha(0,\theta_{-i})) \leq \sum_{i=1}^n c_i(K) = c(K)$, and with a monotone decision rule can be no more than $c(\alpha(\theta))$, since $\sum_{i=1}^n c_i(\alpha(0,\theta_{-i})) \leq \sum_{i=1}^n c_i(\alpha(\theta)) = c(\alpha(\theta))$. Hence, the total CSP payment can be no more than $c(\alpha(\theta)) + c(K)$ and with a monotone decision rule, no more than $2c(\alpha(\theta))$. ■

Let $\mathbb{E}_B \subset \mathbb{E}_F$ denote the set of binary environments, i.e., the set of finite environments with $K = 1$.

Lemma 2. *Given any sequence of binary environments $(\theta^n, z^n, \mathcal{E}^n)_{n\in\mathbb{N}}$ generated by i.i.d. draws from some $F \in \Delta(\Theta_0 \times Z_0)$, the probability that there is at least one pivotal player in $\mathcal{E}^n$ given θ^n goes to zero as n goes to infinity.*

Proof. Let $c_n = c^n(1)$, $\mu = \mathbb{E}(\theta_i)$, $\sigma^2 = \mathrm{Var}(\theta_i)$, and $\theta_n^* = \max_{i\in\{1,\dots,n\}} \theta_i$. Then $\mathbb{P}(k \text{ is pivotal in } \mathcal{E}^n \text{ given } \theta^n) \leq \mathbb{P}(\sum_{j\neq k,\, j\leq n} \theta_j \leq c_n \text{ and } \sum_{i=1}^n \theta_i \geq c_n)$ and

$$\begin{aligned}
\mathbb{P}\left(\exists \text{ pivotal player in } \mathcal{E}^n \text{ given } \theta^n\right) &\leq \mathbb{P}\Big(\exists k : \sum_{j\neq k,\, j\leq n} \theta_j \leq c_n \;\text{ and }\; \sum_{i=1}^n \theta_i \geq c_n\Big)\\
&= \mathbb{P}\Big(\exists k : 0 \leq c_n - \sum_{j\neq k,\, j\leq n} \theta_j \;\text{ and }\; \theta_k \geq c_n - \sum_{j\neq k,\, j\leq n} \theta_j\Big)\\
&\leq \mathbb{P}\Big(\exists k : \theta_k \geq \Big|\sum_{j\neq k,\, j\leq n} \theta_j - c_n\Big|\Big)\\
&\leq \mathbb{P}\Big(\exists k : 2\theta_k \geq \Big|\sum_{i=1}^n \theta_i - c_n\Big|\Big)\\
&= \mathbb{P}\Big(\theta_n^* \geq \frac{1}{2}\Big|\sum_{i=1}^n \theta_i - c_n\Big|\Big).
\end{aligned}$$

We would like to show $\mathbb{P}\Big(\theta_n^* \geq \frac{1}{2}\Big|\sum_{i=1}^n \theta_i - c_n\Big|\Big) \to 0$ as $n \to \infty$. Fix any $\varepsilon > 0$.

$$\begin{aligned}
&\mathbb{P}\Big(\theta_n^* \geq \frac{1}{2}\Big|\sum_{i=1}^n \theta_i - c_n\Big|\Big)\\
&= \mathbb{P}\Big(\frac{\theta_n^*}{\sigma\sqrt{n}} \geq \frac{1}{2}\Big|\frac{\sum_{i=1}^n(\theta_i-\mu)}{\sigma\sqrt{n}} - \frac{c_n - \mu n}{\sigma\sqrt{n}}\Big|\Big)\\
&= \mathbb{P}\Big(\frac{\theta_n^*}{\sigma\sqrt{n}} \geq \frac{1}{2}\Big|\frac{\sum_{i=1}^n(\theta_i-\mu)}{\sigma\sqrt{n}} - \frac{c_n - \mu n}{\sigma\sqrt{n}}\Big|,\ \frac{1}{2}\Big|\frac{\sum_{i=1}^n(\theta_i-\mu)}{\sigma\sqrt{n}} - \frac{c_n - \mu n}{\sigma\sqrt{n}}\Big| > \frac{\varepsilon}{2}\Big)\\
&\qquad + \mathbb{P}\Big(\frac{\theta_n^*}{\sigma\sqrt{n}} \geq \frac{1}{2}\Big|\frac{\sum_{i=1}^n(\theta_i-\mu)}{\sigma\sqrt{n}} - \frac{c_n - \mu n}{\sigma\sqrt{n}}\Big|,\ \frac{1}{2}\Big|\frac{\sum_{i=1}^n(\theta_i-\mu)}{\sigma\sqrt{n}} - \frac{c_n - \mu n}{\sigma\sqrt{n}}\Big| \leq \frac{\varepsilon}{2}\Big)\\
&\leq \mathbb{P}\Big(\frac{\theta_n^*}{\sigma\sqrt{n}} > \frac{\varepsilon}{2}\Big) + \mathbb{P}\Big(\Big|\frac{\sum_{i=1}^n(\theta_i-\mu)}{\sigma\sqrt{n}} - \frac{c_n - \mu n}{\sigma\sqrt{n}}\Big| \leq \varepsilon\Big)\\
&= \mathbb{P}\Big(\frac{\theta_n^*}{\sigma\sqrt{n}} > \frac{\varepsilon}{2}\Big) + \mathbb{P}\Big(\frac{\sum_{i=1}^n(\theta_i-\mu)}{\sigma\sqrt{n}} \leq \frac{c_n - \mu n}{\sigma\sqrt{n}} + \varepsilon\Big) - \mathbb{P}\Big(\frac{\sum_{i=1}^n(\theta_i-\mu)}{\sigma\sqrt{n}} < \frac{c_n - \mu n}{\sigma\sqrt{n}} - \varepsilon\Big)\\
&\equiv \mathbb{P}\Big(\frac{\theta_n^*}{\sigma\sqrt{n}} > \frac{\varepsilon}{2}\Big) + F_n\Big(\frac{c_n - \mu n}{\sigma\sqrt{n}} + \varepsilon\Big) - F_n\Big(\frac{c_n - \mu n}{\sigma\sqrt{n}} - \varepsilon\Big),
\end{aligned}$$

where F_n is the cdf of $\frac{\sum_{i=1}^n(\theta_i-\mu)}{\sigma\sqrt{n}}$. Now, $\frac{\theta_n^*}{\sigma\sqrt{n}} \xrightarrow{p} 0$ by Rob (1982, pp. 211-212, proof of Lemma 1 and 2) and $\frac{\sum_{i=1}^n(\theta_i-\mu)}{\sigma\sqrt{n}} \xrightarrow{d} N(0,1)$ by the central limit theorem. By van der Vaart

(1998, p. 12, Lemma 2.11), if a sequence of random variables X_n with cdf G_n converges in distribution to a random variable X with cdf G and G is continuous, then $G_n \to G$ uniformly. In particular, for any $\delta > 0$ there exists $N(\delta)$ such that for all $n > N(\delta)$ and $x \in \mathbb{R}$, $|G_n(x) - G(x)| < \delta$.

Let Φ denote the cdf of a standard normal, which is continuous. Then $F_n \to \Phi$ uniformly and for any $x \in \mathbb{R}$ and $n > N(\delta)$,

$$\begin{aligned}\Big|\big(F_n(x+\varepsilon) - F_n(x-\varepsilon)\big) - \big(\Phi(x+\varepsilon) - \Phi(x-\varepsilon)\big)\Big| &= \Big|\big(F_n(x+\varepsilon) - \Phi(x+\varepsilon)\big) + \big(\Phi(x-\varepsilon) - F_n(x-\varepsilon)\big)\Big| \\ &\leq |F_n(x+\varepsilon) - \Phi(x+\varepsilon)| + |F_n(x) - \Phi(x)| \\ &< 2\delta.\end{aligned}$$

Since $\max_x \Phi(x+\varepsilon) - \Phi(x-\varepsilon) = \Phi(\varepsilon) - \Phi(-\varepsilon)$,

$$\limsup_{n\to\infty} \mathbb{P}\Big(\theta_n^* \geq \frac{1}{2}\Big|\sum_{i=1}^n \theta_i - c_n\Big|\Big) \leq \Phi(\varepsilon) - \Phi(-\varepsilon),$$

and since $\Phi(\varepsilon) - \Phi(-\varepsilon) \to 0$ as $\varepsilon \to 0$, the result follows. ■

Proposition 6. *Given any sequence of finite environments $(\theta^n, z^n, \mathcal{E}^n)_{n\in\mathbb{N}}$ generated by i.i.d. draws from some $F \in \Delta(\Theta_0 \times Z_0)$, the probability that there is at least one pivotal player in $\mathcal{E}^n$ given θ^n goes to zero as n goes to infinity.*

Proof. Let $\mathcal{E}^n(Y') \in \mathbb{E}_B$ be the environment $\mathcal{E}^n \in \mathbb{E}_F$ with $Y' \subseteq Y$ substituted for Y. For any $i \in I$ and $a, b \in Y$ with $a < b$, let $\theta_i[a,b] \equiv \theta_i(b) - \theta_i(a)$ and $c^n[a,b] \equiv c^n(b) - c^n(a)$. Note that $(\theta_i[a,b])_{i\in\mathbb{N}}$ is i.i.d. and $\mathbb{E}[(\theta_i[a,b])^2] < \infty$. By Lemma 2,

$$\begin{aligned}&\mathbb{P}(\exists \text{ pivotal player in } \mathcal{E}^n(\{a,b\}) \text{ given } \theta^n) \\ &\quad \leq \mathbb{P}\Big(\exists k : \sum_{j\neq k,\, j\leq n} \theta_j[a,b] < c^n[a,b] \ \text{ and } \ \sum_{i=1}^n \theta_i[a,b] \geq c^n[a,b]\Big) \to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

Let $\mathcal{Y} = \{\{a,b\} : a, b \in Y \text{ and } a < b\}$. Note that $|\mathcal{Y}| = K(K+1)/2$. Then

$$\begin{aligned}\mathbb{P}\left(\exists \text{ pivotal player in } \mathcal{E}^n \text{ given } \theta^n\right) &\leq \mathbb{P}\left(\exists Y' \in \mathcal{Y} : \exists \text{ pivotal player in } \mathcal{E}^n(Y') \text{ given } \theta^n\right) \\ &\leq \sum_{Y'\in\mathcal{Y}} \mathbb{P}\left(\exists \text{ pivotal player in } \mathcal{E}^n(Y') \text{ given } \theta^n\right) \\ &\to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

■

Theorem 5. *Given any sequence of finite environments $(\theta^n, z^n, \mathcal{E}^n)_{n\in\mathbb{N}}$ generated by i.i.d. draws from some $F \in \Delta(\Theta_0 \times Z_0)$, any sequence of cost-sharing pivotal mechanisms $((\alpha^n, \tau^n))_{n\in\mathbb{N}}$ is asymptotically ex-post budget-balanced.*

Proof. **Part I.** The probability that no agent is pivotal goes to one by Proposition 6, and the CSP is EPBB when no agent is pivotal.

Part II. Let $S^n(\theta^n) \equiv \sum_{i=1}^n \tau_i^n(\theta^n) - c^n(\alpha^n(\theta^n))$ be the budget surplus of the CSP mechanism (α^n, τ^n). Let $\Pi^n \subset \Theta^n$ be the event in which there exists a pivotal player in $\mathcal{E}^n$ given θ^n.

Case 1. Suppose $\frac{c^n(K)}{n} \not\to \infty$ as $n \to \infty$. Then

$$\begin{aligned}
\frac{1}{n}\mathbb{E}\Big(\big|S^n(\theta^n)\big|\Big) &= \frac{1}{n}\mathbb{E}\Big(\big|S^n(\theta^n)\big| \mid \Pi^n\Big) \cdot \mathbb{P}(\Pi^n) + \frac{1}{n}\mathbb{E}\Big(\big|S^n(\theta^n)\big| \mid \neg\Pi^n\Big) \cdot \mathbb{P}(\neg\Pi^n) \\
&\leq \frac{c^n(K)}{n} \cdot \mathbb{P}(\Pi^n) \\
&\to 0 \quad \text{as} \quad n \to \infty,
\end{aligned}$$

where the inequality follows by Proposition 5 and the last line follows by Proposition 6.

Case 2. Suppose $\frac{c^n(K)}{n} \to \infty$ as $n \to \infty$. Let k^* be the smallest k such that $\frac{c^n(k)}{n} \to \infty$ as $n \to \infty$. Since $c^n(y)$ is non-decreasing in y for all n, $\frac{c^n(k)}{n} \to \infty$ for any $k \geq k^*$. Let $\bar{\theta}^n(k) \equiv \sum_{i=1}^n \theta_i(k)$, $\mu(k) = \mathbb{E}(\theta_i(k))$, and $\sigma^2(k) = \mathrm{Var}(\theta_i(k))$. If $\bar{\theta}^n(k) < c^n(k)$ for all $k \geq \hat{k}$, then the total fair share payment $\sum_{i=1}^n c_i(\alpha(0, \theta_{-i}))$ can be no more than $c(\hat{k}-1)$, since any $k \geq \hat{k}$ is never an efficient decision without i for any i. Hence, the total CSP payment can be no less than zero and no more than $c(\alpha(\theta)) + c(\hat{k}-1)$ by Proposition 5, and the distance from EPBB is bounded by $c(\hat{k}-1)$. We may write

$$\begin{aligned}
\frac{1}{n}\mathbb{E}\Big(\big|S^n(\theta^n)\big|\Big) &= \frac{1}{n}\mathbb{E}\Big(\big|S^n(\theta^n)\big| \mid \forall k \geq k^*,\ \bar{\theta}^n(k) < c^n(k)\Big)\mathbb{P}\Big(\forall k \geq k^*,\ \bar{\theta}^n(k) < c^n(k)\Big) \\
&\quad + \frac{1}{n}\mathbb{E}\Big(\big|S^n(\theta^n)\big| \mid \exists k \geq k^*,\ \bar{\theta}^n(k) \geq c^n(k)\Big)\mathbb{P}\Big(\exists k \geq k^*,\ \bar{\theta}^n(k) \geq c^n(k)\Big).
\end{aligned}$$

Now,

$$\begin{aligned}
&\frac{1}{n}\mathbb{E}\Big(\big|S^n(\theta^n)\big| \mid \forall k \geq k^*,\ \bar{\theta}^n(k) < c^n(k)\Big) \\
&\quad = \frac{1}{n}\mathbb{E}\Big(\big|S^n(\theta^n)\big| \mid \forall k \geq k^*,\ \bar{\theta}^n(k) < c^n(k),\ \Pi^n\Big) \cdot \mathbb{P}(\Pi^n \mid \forall k \geq k^*,\ \bar{\theta}^n(k) < c^n(k)) \\
&\qquad + \frac{1}{n}\mathbb{E}\Big(\big|S^n(\theta^n)\big| \mid \forall k \geq k^*,\ \bar{\theta}^n(k) < c^n(k),\ \neg\Pi^n\Big) \cdot \mathbb{P}(\neg\Pi^n \mid \forall k \geq k^*,\ \bar{\theta}^n(k) < c^n(k)) \\
&\quad \leq \frac{c^n(k^*-1)}{n} \cdot \frac{\mathbb{P}(\Pi^n)}{\mathbb{P}(\forall k \geq k^*,\ \bar{\theta}^n(k) < c^n(k))} \\
&\quad \to 0 \quad \text{as} \quad n \to \infty,
\end{aligned}$$

where the inequality follows by Proposition 5 and the last line follows by Proposition 6 and

$$\mathbb{P}\Big(\exists k \geq k^*,\ \bar{\theta}^n(k) \geq c^n(k)\Big) \leq \sum_{k=k^*}^{K} \mathbb{P}\Big(\frac{1}{n}\sum_{i=1}^n \theta_i(k) \geq \frac{c^n(k)}{n}\Big) \to 0,$$

since for all $k \geq k^*$, $\frac{1}{n}\sum_{i=1}^n \theta_i(k) \xrightarrow{p} \mu(k)$ by the law of large numbers and $\frac{c^n(k)}{n} \to \infty$ by assumption. Moreover,

$$\begin{aligned}
&\frac{1}{n}\mathbb{E}\Big(\big|S^n(\theta^n)\big| \mid \exists k \geq k^*,\ \bar{\theta}^n(k) \geq c^n(k)\Big)\mathbb{P}\Big(\exists k \geq k^*,\ \bar{\theta}^n(k) \geq c^n(k)\Big) \\
&\quad= \frac{1}{n}\mathbb{E}\Big(\big|S^n(\theta^n)\big| \mid \bar{\theta}^n(k^*) \geq c^n(k^*),\ \bar{\theta}^n(k^*+1) < c^n(k^*+1),\ \ldots,\ \bar{\theta}^n(K) < c^n(K)\Big) \\
&\qquad\quad \times \mathbb{P}\Big(\bar{\theta}^n(k^*) \geq c^n(k^*),\ \bar{\theta}^n(k^*+1) < c^n(k^*+1),\ \ldots,\ \bar{\theta}^n(K) < c^n(K)\Big) \\
&\qquad + \frac{1}{n}\mathbb{E}\Big(\big|S^n(\theta^n)\big| \mid \bar{\theta}^n(k^*+1) \geq c^n(k^*+1),\ \bar{\theta}^n(k^*+2) < c^n(k^*+2),\ \ldots,\ \bar{\theta}^n(K) < c^n(K)\Big) \\
&\qquad\quad \times \mathbb{P}\Big(\bar{\theta}^n(k^*+1) \geq c^n(k^*+1),\ \bar{\theta}^n(k^*+2) < c^n(k^*+2),\ \ldots,\ \bar{\theta}^n(K) < c^n(K)\Big) \\
&\qquad + \ldots \\
&\qquad + \frac{1}{n}\mathbb{E}\Big(\big|S^n(\theta^n)\big| \mid \bar{\theta}^n(K) \geq c^n(K)\Big) \times \mathbb{P}\Big(\bar{\theta}^n(K) \geq c^n(K)\Big) \\
&\quad\leq \frac{c^n(k^*)}{n}\mathbb{P}\Big(\bar{\theta}^n(k^*) \geq c^n(k^*)\Big) + \frac{c^n(k^*+1)}{n}\mathbb{P}\Big(\bar{\theta}^n(k^*+1) \geq c^n(k^*+1)\Big) + \ldots + \frac{c^n(K)}{n}\mathbb{P}\Big(\bar{\theta}^n(K) \geq c^n(K)\Big).
\end{aligned}$$

For any $k = k^*, \ldots, K$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $\frac{c^n(k)}{n} > \mu(k)$ and

$$\begin{aligned}
\frac{c^n(k)}{n} \cdot \mathbb{P}\Big(\bar{\theta}^n(k) \geq c^n(k)\Big) &\leq \frac{c^n(k)}{n} \cdot \mathbb{P}\Big(\big|\bar{\theta}^n(k) - n\mu(k)\big| \geq \big|c^n(k) - n\mu(k)\big|\Big) \\
&\leq \frac{c^n(k)}{n} \cdot \frac{n\sigma^2(k)}{\big(c^n(k) - n\mu(k)\big)^2} \\
&= \frac{c^n(k)}{n} \cdot \frac{\sigma^2(k)}{n\big(\frac{c^n(k)}{n} - \mu(k)\big)^2} \\
&\to 0 \quad \text{as} \quad n \to \infty,
\end{aligned}$$

where the second inequality follows by Chebyshev's inequality. ■

Theorem 6. *Consider any sequence of finite environments $(\theta^n, z^n, \mathcal{E}^n)_{n\in\mathbb{N}}$ generated by i.i.d. draws from some $F \in \Delta(\Theta_0 \times Z_0)$. Let $(\alpha^n)_{n\in\mathbb{N}}$ be a sequence of efficient and monotone decision rules and $(\tau^n)_{n\in\mathbb{N}}$ be a sequence of CSP transfer rules. A sequence of efficient and monotone mechanisms $((\alpha^n, \hat{\tau}^n))_{n\in\mathbb{N}}$ satisfies AEPBB and strategy-proofness and cost-sharing universal participation for each n if and only if, for all i,*

$$\hat{\tau}_i^n(\theta^n) = \tau_i^n(\theta) + h_i^n(\theta_{-i}^n),$$

for some sequence of functions $h_i^n : \Theta_{-i} \to \mathbb{R}$ such that

1. *$h_i^n(\theta_{-i}^n) \leq 0$ for all $\theta_{-i}^n \in \Theta_{-i}^n$, i, and n,*
2. *$\lim_{n\to\infty} \mathbb{P}(h_i^n(\theta_{-i}^n) = 0$ for all $i) = 1$, and*
3. *$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \mathbb{E}\left[h_i^n(\theta_{-i}^n)\right] = 0$.*

The same holds replacing cost-sharing universal participation with the fair pricing principle per unit. Under Assumption 1, the same holds replacing the fair pricing principle per unit with the fair pricing principle.

Proof. **Part I.** WTS $(\alpha^n, \hat{\tau}^n)$ satisfies strategy-proofness and CS-UP (the FPP per unit, and the FPP under Assumption 1) for each n if and only if (1). This follows immediately from Holmström (1979) and Theorem 1 (Theorem 3).

Part II. WTS $(\alpha^n, \hat{\tau}^n)$ satisfies strategy-proofness, CS-UP (the FPP per unit, and the FPP under Assumption 1) for each n, and AEPBB if and only if (1), (2), and (3).

$$\begin{aligned}
&\mathbb{P}\Big(\sum_{i=1}^{n} \hat{\tau}_i^n(\theta^n) - c^n(\alpha^n(\theta^n)) = 0\Big) \\
&= \mathbb{P}\Big(\sum_{i=1}^{n} \hat{\tau}_i^n(\theta^n) - c^n(\alpha^n(\theta^n) = 0 \mid \neg\exists \text{ pivotal player in } \mathcal{E}^n \text{ given } \theta^n\Big)\mathbb{P}(\neg\exists \text{ pivotal player in } \mathcal{E}^n \text{ given } \theta^n) \\
&\quad + \mathbb{P}\Big(\sum_{i=1}^{n} \hat{\tau}_i^n(\theta^n) - c^n(\alpha^n(\theta^n) = 0 \mid \exists \text{ pivotal player in } \mathcal{E}^n \text{ given } \theta^n\Big)\mathbb{P}(\exists \text{ pivotal player in } \mathcal{E}^n \text{ given } \theta^n) \\
&\equiv \mathbb{P}(\text{EPBB} \mid \text{no one pivotal})\mathbb{P}(\text{no one pivotal}) + \mathbb{P}(\text{EPBB} \mid \text{someone pivotal})\mathbb{P}(\text{someone pivotal}).
\end{aligned}$$

Hence,

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}(\text{EPBB}) &= \lim_{n\to\infty} \mathbb{P}(\text{EPBB} \mid \text{no one pivotal}) \lim_{n\to\infty} \mathbb{P}(\text{no one pivotal}) \\
&\quad + \lim_{n\to\infty} \mathbb{P}(\text{EPBB} \mid \text{someone pivotal}) \lim_{n\to\infty} \mathbb{P}(\text{someone pivotal}) \\
&= \lim_{n\to\infty} \mathbb{P}(\text{EPBB} \mid \text{no one pivotal}) \\
&= \lim_{n\to\infty} \mathbb{P}\Big(\sum_{i=1}^{n} h_i^n(\theta_{-i}^n) = 0 \mid \text{no one pivotal}\Big) \\
&= \lim_{n\to\infty} \mathbb{P}(h_i^n(\theta_{-i}^n) = 0 \ \forall i \mid \text{no one pivotal}) \\
&= \lim_{n\to\infty} \mathbb{P}(h_i^n(\theta_{-i}^n) = 0 \ \forall i),
\end{aligned}$$

where the second line follows since $\lim_{n\to\infty} \mathbb{P}(\text{someone pivotal}) = 0$ (proof of Theorem 5 Part I), the third line follows since a CSP is EPBB when no one is pivotal, the fourth line follows from Part I, and the last line follows since

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}(h_i^n(\theta_{-i}^n) = 0 \ \forall i) &= \lim_{n\to\infty} \mathbb{P}(h_i^n(\theta_{-i}^n) = 0 \ \forall i \mid \text{no one pivotal}) \lim_{n\to\infty} \mathbb{P}(\text{no one pivotal}) \\
&\quad + \lim_{n\to\infty} \mathbb{P}(h_i^n(\theta_{-i}^n) = 0 \ \forall i \mid \text{someone pivotal}) \lim_{n\to\infty} \mathbb{P}(\text{someone pivotal}) \\
&= \lim_{n\to\infty} \mathbb{P}(h_i^n(\theta_{-i}^n) = 0 \ \forall i \mid \text{no one pivotal}).
\end{aligned}$$

Finally,

$$
\begin{aligned}
& \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[\left|\sum_{i=1}^{n} \tau_i^n(\theta^n) + h_i^n(\theta_{-i}^n) - c^n(\alpha^n(\theta^n))\right|\right] = 0 \\
\iff & \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[\sum_{i=1}^{n} \tau_i^n(\theta^n) - c^n(\alpha^n(\theta^n))\right] + \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\left[h_i^n(\theta_{-i}^n)\right] = 0
\end{aligned}
$$

where the second line follows since $\lim_{x\to\infty} f(x) = 0 \iff \lim_{x\to\infty}|f(x)| = 0$[46] and the desired result follows by Theorem 5. ■

[46]Notice $|f(x) - 0| < \varepsilon \iff ||f(x)| - 0| < \varepsilon$.