Public Good Provision with No Extortion

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Preliminary Draft
November 5, 2019

(Latest Version Here)

Abstract

I consider a classic public good provision problem when the government has the power to tax its citizens. In this environment, participation constraints need not be satisfied. I replace such participation constraints with a weaker condition, which I call *no-extortion*, that limits the ability of the government to extract funds from its citizens. It is well known that there does not exist any strategy-proof, efficient, and budget-balanced mechanism. In fact, any strategy-proof and efficient mechanism that additionally satisfies individual-rationality or universal-participation fails to raise *any* revenue in large populations. However, replacing these conditions with no-extortion yields a positive result. There exists a simple, detail-free mechanism that is strategy-proof, efficient, extortion-free, and (asymptotically) budget-balanced in large populations. Furthermore, among all strategy-proof, efficient, and extortion-free mechanisms, this mechanism is undominated and uniquely maximizes ex-post revenue (minimizing any potential, though unlikely, budget deficit).

1 Introduction

Consider a large group of individuals who would like to decide whether a costly non-excludable public good is worth providing to the community. That is, they wish to answer the question: is the sum of each individual's value for the public good at least as high as the cost of providing it? Each individual knows only her *own* value, so in order to compute the community's total valuation, everyone must be willing to truthfully reveal their private information. Mechanism design provides a rigorous framework to analyze such problems. A mechanism is a mapping from

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all individuals' reports of their private information to social outcomes, and it is feasible (or incentive-compatible) if it is an equilibrium for each individual to report their private information truthfully. In this paper, I focus on the robust solution concept of *strategy-proofness*, which says that truthful reporting is a dominant-strategy equilibrium.

A social outcome consists of a selection from a set of social alternatives, in this case a binary decision to provide or not provide the public good, along with a specification of payments for each individual in the community. In principle, a mechanism in this setting cannot feasibly result in a budget imbalance, i.e. payments can never sum to less (budget deficit) or more (budget surplus) than the implementation cost of the chosen alternative. In the case of a deficit, it is not physically possible to implement the chosen alternative without raising the funds to do so, which, barring donations from a third-party, must come from within the community. In the case of a surplus, while it is physically possible to "burn" the money or donate it to a third-party, this is likely not acceptable in practice, and hence the funds must ultimately be redistributed back to the community. But such taxes and rebates are, in principle, a part of each individual's transfer payment, closing the system and eliminating the possibility of budget imbalance in the first place. That being said, this paper will take the view that a government can finance *small* deficits and redistribute small surpluses with *imperceptible* changes to its tax policy. That is, we will accept violations of budget-balance in either direction as feasible, as long as they are "small", with the understanding that the resulting budget-balancing taxes or rebates will be sufficiently insignificant and indirect so that individuals do not perceive them as part of their transfer payments.[1] I refer to this as *approximate budget-balance*.

It is common in mechanism design to also require some form of participation constraint. This captures the idea that, while it may be a dominant strategy for an individual to report truthfully conditional on participating in the mechanism, they may have good reason to avoid participating in the first place. A key argument of this paper is that, in the context of a public good provision problem with a government that has the power to tax its citizens, such participation constraints are *not* necessary. Consider an individual who refuses to "participate". This individual will still derive value from the choice (made in her stead) to provide or not provide the public good, since public goods are by assumption not excludable.[2] She will also be

[1] Practically, this can be understood as a government savings account capable of absorbing small deficits and surpluses. Over time, any accumulated surplus will be repaid to the community through tax cuts, while any accumulated deficit will be replenished through tax increases. If these tax adjustments are small, we assume individuals do not perceive them as part of the original mechanism.

[2] Many public goods may be excludable in principle but not in practice. For example, in principle it is possible to build a fence around a park and hire a security guard to attend the entrance gate. However, in practice this may be prohibitively expensive and may also detract from the citizens'

unable to avoid her transfer payment, since by assumption the government has the power to tax.[3]

But this is not to say that constraints in this style have no use here. While I have argued that they should not be interpreted as providing incentives to "participate", these constraints can often be interpreted as providing conditions for "fairness". For example, ex-post individual-rationality (EPIR) provides an especially strong notion of fairness—that each individual is happier after the mechanism concludes than before it began. As we will see, such a strong notion of fairness will be impossible to satisfy alongside other crucial properties. Hence, I define a weaker fairness condition that seems quite natural in the context of public goods provision—that each individual can only be asked to pay more than her value of the chosen alternative if this amount does not exceed her fair share λ_i of its implementation cost.[4] This captures the idea that individuals are willing to pay up to their value of the good, but also find it acceptable if they are asked to contribute up to their fair share of its implementation cost. I call this condition *no-extortion*.

As alluded to in the first paragraph, the final property we seek is efficiency. That is, we will focus on decision rules that provide the public good if and only if the sum of values is at least as high as the cost of providing it. These are known as *efficient* decision rules.

In summary, we seek a mechanism that is strategy-proof, approximately budget-balanced, extortion-free, and efficient. Such a mechanism does exist, and I provide one in particular that is simple, intuitive, and detail-free.[5] I call this the *cost-sharing pivot mechanism*, and show that, while it is not the only mechanism that satisfies these four properties, it is undominated[6] and uniquely minimizes any budget deficit among such mechanisms. Intuitively, the cost-sharing pivot is simply a standard pivot mechanism in which the transfers for player i are increased by her share of the implementation cost of what would have been provided without her.

section 2 reviews the literature on the public good provision problem and the difficul-

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experience of the park itself. For analysis of excludable public goods, see Deb, Ghosh and Seo (2002) and Massó et al. (2015).

[3] Alternatively, we might say that participation constraints are still necessary, but not in their standard form. In particular, an individual can move to another community, avoiding any public goods and taxes from her original community. But, given that we will bound how much the government can tax its citizens via a fairness constraint, this is likely to be sufficiently costly so as to never bind.

[4] What constitutes "fair" is agreed upon prior to the mechanism. A simple example is equal shares $\lambda_i = 1/n$, but we may also use exogenous data to inform more sophisticated notions of fairness, e.g. equal shares of one's wealth.

[5] A mechanism is detail-free if it does not need to be calibrated by knowledge of player value distributions or player beliefs.

[6] There is no other strategy-proof, approximately budget-balanced, extortion-free, and efficient mechanism that always gets closer to budget balance.

ties it presents. section 3 introduces the general model and defines the cost-sharing pivot mechanism. section 4 states the main results, and section 5 concludes.

2 The Public Good Provision Problem

The mechanism design approach to the public good provision problem can be understood as a search for mechanisms with sufficiently desirable properties. At first pass, many would agree that the following four axioms come to mind: strategy-proofness, efficiency, ex-post individual-rationality, and ex-post budget-balance. A mechanism is strategy-proof (SP) if it is a dominant-strategy for every participant to report truthfully, efficient (E) if the good is provided if and only if the sum of values exceeds its implementation cost, ex-post individually-rational (EPIR) if every participant is better off after the mechanism concludes than before it began, and ex-post budget-balanced (EPBB) if the transfer payments always sum to exactly the implementation cost. However, this leads to an immediate dead end, as no such mechanism exists. In particular, Green and Laffont (1979) show that no mechanism can satisfy SP, E, and EPBB.

From here, we may proceed to relax certain axioms, hoping that a slightly less ideal, but still satisfactory, mechanism will be unveiled. There are two ways to justify such relaxations. The first is to argue that an existing axiom is unnecessary for a certain context of interest, and that we can hence remove or weaken it "for free" (my justification for weakening EPIR). The second is to argue that we may relax an axiom from holding exactly to holding "approximately" (my justification for weakening EPBB).

Although this paper is interested in strategy-proof and detail-free implementation, let me begin by mentioning two classic papers that do not impose such robustness. d'Aspremont and Gérard-Varet (1979) show that it *is* possible to achieve Bayesian incentive-compatibility (BIC), E, and EPBB, while Myerson and Satterthwaite (1983) show that it is *not* possible to achieve BIC, E, EPBB, and interim individual-rationality (IIR). The latter is an alternative way to see the impossibility of our first-pass axioms.

Now, let us consider relaxing E in hopes that it can be made to hold approximately. Kuzmics and Steg (2017) show that the welfare-maximizing mechanism among all SP, EPIR, and ex-post budget-surplus (EPBS)[7] mechanisms is a "split-the-cost" mechanism, in which each player has a fixed cost share, the good is provided if and only if all players' values exceed their own cost share, and each player pays her cost share if the good is provided and zero otherwise. Notice that this mechanism is in fact EPBB, but is *not* approximately efficient—its performance substantially

[7]A mechanism is EPBS if it never runs a deficit.

deteriorates with n. In fact, Mailath and Postlewaite (1990) show that this asymptotic inefficiency holds for any BIC, IIR, and ex-ante budget-balanced (EABB)[8] mechanism.

Next, consider weakening EPBB to EPBS. I show that no mechanism can satisfy SP, E, EPIR, and EPBS (Corollary 1 in Appendix A). Define no-arbitrage (NA) as collecting no payments from players with weakly positive values for the good when the good is *not* provided. This is a minimal standard of fairness, preventing a government from extracting meaningless funds (since no project is provided) from citizens who are not getting their preferred outcome. I show that no mechanism can satisfy SP, E, NA, and EPBS either (Proposition 2 in Appendix A).

Finally, consider relaxing EPBB in hopes that it can be made to hold approximately. The revenue-maximizing mechanism among all SP, E, and EPIR mechanisms fails to raise *any* revenue in large populations when the good is provided.[9] However, weakening EPIR to no-extortion (NE) yields a positive result—the main result of this paper. The revenue-maximizing mechanism among all SP, E, and NE mechanisms is the cost-sharing pivot, which *is* approximately EPBB. In particular, the probability that it is *not* EPBB and the expected *distance* from EPBB go to zero as the population size goes to infinity.

3 The Cost-Sharing Pivot Mechanism

Let Y be an arbitrary set of social alternatives and $C : Y \to \mathbb{R}$ represent the implementation costs of each alternative.[10] Let $I = \{1, \ldots, n\}$ be a set of individuals and Θ_i be an arbitrary type space for each i. Define $X \equiv Y \times \mathbb{R}^n$ to be the set of final outcomes, which combines a social alternative with a vector of transfers. Each individual i has preferences over X represented by the quasilinear utility function

$$u_i(x, \theta) = v_i(y, \theta_i) - t_i,$$

where $\theta_i \in \Theta_i$ is i's privately known type and t_i is the amount i is asked to pay. Notice that i's value for each alternative v_i does not depend on the private information of i's opponents. This is known as a *private values* assumption.

[8]A mechanism is EABB if it is budget-balanced in expectation, given some belief over value profiles.

[9]With non-negative values, the revenue-maximizing mechanism among all SP, E and EPIR mechanisms is a pivot mechanism, which raises asymptotically zero revenue. With no lower bound on values, the revenue-maximizing mechanism raises less revenue still.

[10]Formally, I model implementation costs using an additional player $i = 0$ called the "implementer" who bears the full implementation cost and is otherwise (commonly known to be) indifferent between all $y \in Y$, so that $u_0(x, \theta_0) = -C(y)$ for all $\theta_0 \in \Theta_0$. For clarity, I remove this "player" from the set I and explicitly write the (negative) implementation costs in place of her value.

A direct revelation mechanism asks each individual to report their type and selects a final outcome as a function of these reports. We seek a direct revelation mechanism that chooses a value-maximizing social alternative given the reports (along with some transfers) and for which it is a dominant strategy for each individual to report truthfully. This focus on direct revelation mechanisms, instead of general mechanisms in which reports can be arbitrary, is without loss of generality due to the *revelation principle* (Gibbard, 1973).

In this environment, a direct revelation mechanism f (henceforth mechanism) can be represented as a pair $f = (\alpha, \tau)$, where $\alpha : \Theta \to Y$ is a decision rule and $\tau : \Theta \to \mathbb{R}^n$ is a transfer rule. Define the efficient decision rule by

$$\alpha(\theta) \in \arg\max_{y \in Y} \sum_{i \in I} v_i(y, \theta_i) - C(y)$$

and the efficient decision rule without i by $\alpha_{-i}(\theta_{-i}) \in \arg\max_{y \in Y} \sum_{j \neq i} v_j(y, \theta_j) - C(y)$.

Definition 1. A mechanism $f = (\alpha, \tau)$ is a *cost-sharing pivot mechanism* given cost shares λ if the decision rule is efficient and the transfer rule satisfies

$$\tau_i(\theta) = \left(\sum_{j \neq i} v_j(\alpha_{-i}(\theta_{-i}), \theta_j) - C(\alpha_{-i}(\theta_{-i})) \right) - \left(\sum_{j \neq i} v_j(\alpha(\theta), \theta_j) - C(\alpha(\theta)) \right) + \lambda_i(\alpha_{-i}(\theta_{-i})) C(\alpha_{-i}(\theta_{-i}))$$

where $\sum_j \lambda_j(y) = 1$ and $\lambda_i(y) \geq 0$ for all i and y.

The $\lambda_i(y)$ are interpreted as i's "fair share" of the implementation costs for alternative y. A simple notion of fair share would be equal shares, i.e. $\lambda_i(y) = \frac{1}{n}$ for all y, but the mechanism does not rely on such symmetry. Indeed, if exogenous data like wealth levels w_i are available, an appealing notion of fair share might be equal shares of one's *wealth*, i.e. $\lambda_i(y) = w_i / \sum_j w_j$ for all y.

Intuition for the cost-sharing pivot is probably best understood through its connection to the class of VCG mechanisms and, in particular, the pivot mechanism.

Definition 2. A mechanism $f = (\alpha, \tau)$ is a *VCG mechanism* if the decision rule is efficient and the transfer rule satisfies

$$\tau_i(\theta) = -\left(\sum_{j \neq i} v_j(\alpha(\theta), \theta_j) - C(\alpha(\theta)) \right) + g_i(\theta_{-i})$$

for any function $g_i : \Theta_{-i} \to \mathbb{R}$, which we will call the *VCG constant* (it is constant in i's report).

These mechanisms, due to Vickrey (1961), Clarke (1971), and Groves (1973), cleverly set each individual's ex-post payoff $u_i(\alpha(\theta), \theta)$ equal to the total social value of $\alpha(\theta)$. Since the decision rule is efficient, this perfectly aligns each individual's objective to the mechanism's, and so it must be a weakly dominant strategy for each individual to report truthfully (strategy-proofness). Furthermore, we may add any term to i's transfer that does not depend on i's report (the VCG constant) without altering these incentives, which completes the class of VCG mechanisms. Remarkably, Green and Laffont (1977) and Holmström (1979) show that, as long as the space of admissible values is sufficiently rich, these are the *only* mechanisms that are strategy-proof and efficient.

Definition 3. A mechanism $f = (\alpha, \tau)$ is a *pivot mechanism* if f is a VCG mechanism with VCG constant

$$g_i(\theta_{-i}) = \sum_{j \neq i} v_j(\alpha_{-i}(\theta_{-i}), \theta_j) - C(\alpha_{-i}(\theta_{-i})).$$

The pivot mechanism is perhaps the most intuitive of the VCG mechanisms. Its transfers are set so that pivotal players (those whose presence changes the decision) compensate the other players for this change, while non-pivotal players pay nothing. This is achieved by adding a constant to i's VCG transfer equal to the total value to i's opponents of the social alternative that would have been chosen without i. The pivot mechanism has several desirable properties in environments with costless alternatives (i.e. when $C(y) = 0$ for all y). In particular, it never runs a budget-deficit and is asymptotically budget-balanced (Green and Laffont, 1979; Rob, 1982).[11] One adaptation of the pivot mechanism that retains these properties in environments with costly alternatives is the following.

Definition 4. A mechanism $f = (\alpha, \tau)$ is a *pivot mechanism with fixed payment plan* if the decision rule is efficient and the transfer rule satisfies

$$\begin{aligned}\tau_i(\theta) = &\sum_{j \neq i} \left[v_j(\tilde{\alpha}_{-i}(\theta_{-i}), \theta_j) - \lambda_j(\tilde{\alpha}_{-i}(\theta_{-i}))C(\tilde{\alpha}_{-i}(\theta_{-i}))\right] \\ &- \sum_{j \neq i} \left[v_j(\alpha(\theta), \theta_j) - \lambda_j(\alpha(\theta))C(\alpha(\theta))\right] + \lambda_i(\alpha(\theta))C(\alpha(\theta)),\end{aligned}$$

where $\sum_j \lambda_j(y) = 1$, $\lambda_i(y) \geq 0$ for all i and y, and

$$\tilde{\alpha}_{-i}(\theta_{-i}) \in \arg\max_{y \in Y} \sum_{j \neq i} \left[v_j(y, \theta_j) - \lambda_j(y)C(y)\right].$$

A bit of staring reveals that this is, in fact, equivalent to a standard pivot mechanism in a modified environment where each outcome is interpreted as including a

[11] Green and Laffont (1979) and Rob (1982) show asymptotic budget-balance for binary decision problems, but their results can easily be generalized to environments with finitely many alternatives.

fixed division of implementation costs for each player. Thus, types in this modified environment $\tilde{\theta}_i$ take into account that i will be forced to pay $\lambda_i(y)C(y)$ if y is implemented, so that $v_j(y, \tilde{\theta}_i) = v_i(y, \theta_i) - \lambda_i(y)C(y)$. This is a standard approach to modeling implementation costs in the literature[12] and has the obvious and powerful benefit that the mathematical analysis between environments with costless and costly alternatives is identical. Indeed, Green and Laffont (1979, p. 31) even contend that "there is no real alternative to this approach", since "the financing decision cannot be effected simultaneously with the elicitation of preferences."

While it is true that the choice of payment plan must be arbitrary in an efficient mechanism,[13] I do not believe that implicitly embedding cost shares inside the alternatives themselves is the only effective *modeling* approach. First, doing so immediately removes the costs from the framework, making it harder to think carefully about designing solutions around them. Indeed, this paper is built upon a careful examination of axioms encompassing explicit costs. Second, not all axioms carry the same interpretation with intrinsic values as with "net values" (values net of one's cost share). This can lead to results that do *not* carry over from environments with costless alternatives to those with costly alternatives and embedded payment plans.

Consider the pivot mechanism and a condition known as universal participation (UP). Suppose that a player may abstain from the mechanism and avoid transfer payments, but will face the outcome of the mechanism regardless. Then UP guarantees that no player can profitably abstain. The pivot mechanism satisfies UP. However, this interpretation does not extend to the pivot mechanism with fixed payment plan, in which players *can* profitably abstain in order to avoid paying their share of the implementation cost. Suppose instead that a player's abstention voids only additional payments made on top of her share of the selected outcome's implementation cost, which she must pay regardless. Only under this qualitatively different interpretation does the pivot mechanism with fixed payment plan guarantee that no player can profitably abstain. In other words, the interpretation of UP changes when replacing intrinsic values with "net values", so care must be taken when drawing parallels between these environments.

The cost-sharing pivot mechanism is an alternative way to adapt the pivot to environments with costly alternatives—one that arises naturally when keeping costs an explicit part of the model. Instead of embedding cost shares within the alternatives and applying the standard pivot, this mechanism adds a "cost-sharing" constant to i's pivot transfer equal to i's fair share of the implementation cost of the social alter-

[12]See for instance Green and Laffont (1979), Moulin (1986), Deb, Ghosh and Seo (2002), Bierbrauer and Hellwig (2016), and Drexl and Kleiner (2018). In fact, Definition 4 is precisely Clarke's (1971) original public goods provision mechanism.

[13]With quasilinear non-altruistic preferences, payment plans are inherently zero-sum, so any efficient decision rule will remain agnostic with respect to the distribution of payments.

native *that would have been chosen without her*. In the next section, we will see that this mechanism satisfies several desirable properties. In particular, the cost-sharing pivot satisfies no-extortion (a natural fairness condition), while the pivot with fixed payment plan does not. Indeed, the pivot with fixed payment plan does not even satisfy no-arbitrage, i.e. it sometimes collects payments from players with a zero or positive value for the good when the good is *not* provided.

4 Binary Decision Environments

We shall now consider in detail the simplest case of a public good provision problem—the case of a binary decision. Let $Y = \{0, 1\}$, where $y = 0$ represents keeping the status quo and $y = 1$ the construction of a public project. It is then convenient to normalize an individual's type to be her value of the project over the status quo and to normalize the implementation cost of the status quo and public project to be zero and c, respectively. Hence,

$$v_i(y, \theta_i) = \theta_i y \quad \text{and} \quad C(y) = \begin{cases} c & \text{if } y = 1 \\ 0 & \text{if } y = 0 \end{cases} .$$

We will also make one important assumption.

Assumption 1 (Non-Negativity)**.** For each individual i, the set of admissible values

$$\Theta_i \in \{[0, b_i] : b_i > 0\} \cup [0, \infty).$$

Assumption 1 rules out the possibility of individuals strictly preferring the status quo to the public project. This may happen, for example, if the construction of a bridge dramatically increases traffic and noise for nearby residents. It turns out that this assumption is crucial to obtain positive results when imposing reasonable fairness conditions. In particular, we will be unable to satisfy both fairness and asymptotic budget-balance if negative values are admissible. See Appendix B for a full discussion.

Given Assumption 1, we may restrict ourselves to the non-trivial case that $c \geq 0$.[14] Denote a binary decision environment with these assumptions and normalizations by $\mathcal{E} = (X, C, I, \{\Theta_i\}_{i \in I}, \{u_i\}_{i \in I}) \in \mathbb{E}_{B_+}$.

We will now consider the following three axioms.

Definition 5. A mechanism f is *strategy-proof* if for all i,

$$u_i(f(\theta_i, \theta_{-i}), \theta_i) \geq u_i(f(\hat{\theta}_i, \theta_{-i}), \theta_i) \quad \forall \hat{\theta}_i,\ \theta.$$

[14] Any project with $c < 0$ would always be provided. Without Assumption 1, this is without loss of generality.

Definition 6. A mechanism $f = (\alpha, \tau)$ is *efficient* if

$$\alpha(\theta) \in \arg\max_{y \in Y} \sum_{i \in I} v_i(y, \theta_i) - C(y).$$

Definition 7. A mechanism $f = (\alpha, \tau)$ is *extortion-free* given cost shares λ if for all i,

$$\tau_i(\theta) \leq \max\{v_i(\alpha(\theta), \theta_i),\ \lambda_i C(\alpha(\theta))\} \quad \forall \theta,$$

where $\lambda_i \geq 0$, and $\sum_j \lambda_j = 1$.

A mechanism is strategy-proof if it is a dominant strategy for each individual to report her type truthfully. A mechanism is efficient if, given truthful reports, the chosen social alternative is utilitarian. A mechanism is extortion-free if each individual can only be asked to pay more than her value of the chosen alternative if this amount does not exceed her fair share λ_i of its implementation cost, where what constitutes "fair" is agreed upon prior to the mechanism. Again, we might consider using equal shares $\lambda_i = \frac{1}{n}$ for all i, or base the shares on exogenously available data $z = (z_1, \dots, z_n)$.

Mathematically, there is a very close connection between no-extortion and ex-post individual-rationality (EPIR). Given our normalization of each individual's value of the status quo to zero, EPIR requires that for all i,

$$\tau_i(\theta) \leq v_i(\alpha(\theta), \theta_i) \quad \forall \theta.$$

This says that each individual must never be worse off after the mechanism concludes than before it began. No-extortion says that an individual may be worse off when the mechanism concludes, but, if this is the case, she cannot be asked to pay more than her fair share of the implementation cost of the chosen alternative.

EPIR is commonly interpreted as giving individuals a dominant-strategy to "participate". However, in a public good provision environment, this interpretation requires that either 1) the public good will not be provided if even a single individual does not participate, or 2) the public good is excludable, so that an individual who does not participate cannot consume the public good when it is provided. The first is not compelling, and the second violates the definition of a (pure) public good. Instead, in this context EPIR should be interpreted simply as a strong fairness condition (no one is made worse off by the mechanism). Through this lens, no-extortion is a weaker fairness condition that embodies an individual's willingness to contribute her "fair share".

In public good environments where the government does *not* have the power to tax, participation constraints are more effectively captured by the concept of universal participation (UP) à la Green and Laffont (1979) (or equivalently no-free-ride à la Moulin (1986)). UP requires that for all i,

$$\tau_i(\theta) \leq v_i(\alpha(\theta), \theta_i) - v_i(\alpha_{-i}(\theta_{-i}), \theta_i) \quad \forall \theta$$

where, as before, $\alpha_{-i}(\theta_{-i})$ is the efficient decision without i. This embodies the idea that if i refuses to participate she will avoid any transfer payments the mechanism would have otherwise imposed, but the mechanism will still operate in her stead and, given the public nature of the decision, she will still "consume" whatever is implemented. In particular, if the public good is provided, she too will benefit.

One might then consider defining no-extortion relative to universal participation by

$$\tau_i(\theta) \leq \max\{v_i(\alpha(\theta), \theta_i) - v_i(\alpha_{-i}(\theta_{-i}), \theta_i),\ \lambda_i C(\alpha(\theta))\} \quad \forall \theta.$$

As a fairness condition, I believe this is less compelling and harder to interpret than Definition 7, but fortunately there is nothing to consider given the following proposition.[15]

Proposition 1. *Given any environment $\mathcal{E} \in \mathbb{E}_{B_+}$, a mechanism f satisfies strategy-proofness and (EPIR-based) no-extortion if and only if it satisfies strategy-proofness and UP-based no-extortion.*

Proof. Suppose $f = (\alpha, \tau)$ is strategy-proof. Then it is well-known that $\tau_i(\theta)$ can only depend on θ_i through $\alpha(\theta)$, i.e. given θ_{-i} all reports $\hat{\theta}_i$ which leave the allocation $\alpha(\hat{\theta}_i, \theta_{-i})$ unchanged must produce the same transfers. First, suppose $\alpha_{-i}(\theta_{-i}) = 0$. Then we immediately have

$$\max\{v_i(\alpha(\theta), \theta_i) - v_i(\alpha_{-i}(\theta_{-i}), \theta_i),\ \lambda_i C(\alpha(\theta))\} = \max\{v_i(\alpha(\theta), \theta_i),\ \lambda_i C(\alpha(\theta))\}$$

since $v_i(0, \theta_i) = 0$. Now, suppose $\alpha_{-i}(\theta_{-i}) = 1$. Then

$$\begin{aligned}
&\tau_i(\theta) \leq \max\{v_i(\alpha(\theta), \theta_i) - v_i(\alpha_{-i}(\theta_{-i}), \theta_i),\ \lambda_i C(\alpha(\theta))\}\ \forall \theta \\
&\iff \tau_i(\theta) \leq \max\{0,\ \lambda_i c\}\ \forall \theta \\
&\iff \tau_i(\theta) \leq \lambda_i c\ \forall \theta
\end{aligned}$$

and

$$\begin{aligned}
&\tau_i(\theta) \leq \max\{v_i(\alpha(\theta), \theta_i),\ \lambda_i C(\alpha(\theta))\}\ \forall \theta \\
&\iff \tau_i(\theta) \leq \max\{\theta_i,\ \lambda_i c\}\ \forall \theta \\
&\iff \tau_i(\theta) \leq \lambda_i c\ \forall \theta
\end{aligned}$$

where the last line follows since $\alpha(\theta_i, \theta_{-i}) = 1$ for every $\theta_i \in \Theta_i$, so strategy-proofness requires that $\tau_i(\theta)$ not depend on θ_i. ■

This brings us to our first main result.

[15] Notice that setting $\lambda_i = 0$ in Proposition 1 implies the equivalence of EPIR and UP given strategy-proofness in these environments as well.

Theorem 1. *Given any environment $\mathcal{E} \in \mathbb{E}_{B_+}$, the unique ex-post revenue-maximizing mechanism $f = (\alpha, \tau)$ among all strategy-proof, efficient, and extortion-free mechanisms is the cost-sharing pivot.*

Proof. Given any environment $\mathcal{E} \in \mathbb{E}_{B_+}$, a mechanism f is strategy-proof and efficient if and only if it is a VCG mechanism by Holmström (1979). Denote VCG transfers by

$$\tau_i(\theta) = \sum_{j \neq i} v_j(\alpha_{-i}(\theta_{-i}), \theta_j) - C(\alpha_{-i}(\theta_{-i})) - \left(\sum_{j \neq i} v_j(\alpha(\theta), \theta_j) - C(\alpha(\theta)) \right) + h_i(\theta_{-i})$$

for any function $h_i : \Theta_{-i} \to \mathbb{R}$. No extortion requires that for all i,

$$\begin{aligned} & \max \{v_i(\alpha(\theta), \theta_i),\ \lambda_i C(\alpha(\theta))\} \geq \tau_i(\theta) \quad \forall \theta \\ \iff & \max \{v_i(\alpha(\theta), \theta_i),\ \lambda_i C(\alpha(\theta))\} - \left(\sum_{j \neq i} v_j(\alpha_{-i}(\theta_{-i}), \theta_j) - C(\alpha_{-i}(\theta_{-i})) \right) \\ & \qquad + \left(\sum_{j \neq i} v_j(\alpha(\theta), \theta_j) - C(\alpha(\theta)) \right) \geq h_i(\theta_{-i}) \quad \forall \theta \end{aligned}$$

In order to maximize ex-post revenue subject to no extortion, we would like to set $h_i(\theta_{-i})$ to the minimum of the LHS with respect to θ_i for each θ_{-i}, i.e.

$$\begin{aligned} h_i(\theta_{-i}) = \min_{\theta_i \in \Theta_i} \Bigg\{ & \max \{v_i(\alpha(\theta), \theta_i),\ \lambda_i C(\alpha(\theta))\} - \left(\sum_{j \neq i} v_j(\alpha_{-i}(\theta_{-i}), \theta_j) - C(\alpha_{-i}(\theta_{-i})) \right) \\ & + \left(\sum_{j \neq i} v_j(\alpha(\theta), \theta_j) - C(\alpha(\theta)) \right) \Bigg\}. \end{aligned}$$

To solve this, let us look for a minimizer. Since the middle term does not depend on θ_i, we can solve

$$\arg\min_{\theta_i \in \Theta_i} \left\{ \max \{v_i(\alpha(\theta), \theta_i),\ \lambda_i C(\alpha(\theta))\} + \sum_{j \neq i} v_j(\alpha(\theta), \theta_j) - C(\alpha(\theta)) \right\}.$$

First, suppose $\alpha_{-i}(\theta_{-i}) = 0$ and let $z = c - \sum_{j \neq i} \theta_j > 0$. If $\theta_i \in [0, z)$, the good is not provided and the objective function is zero. If $\theta_i \in [z, \infty)$, the good is provided and the objective function is non-negative and (weakly) increasing in θ_i.

Now, suppose $\alpha_{-i}(\theta_{-i}) = 1$. If $\theta_i \in [0, \infty)$, the good is provided and the objective function is non-negative and (weakly) increasing in θ_i.

Hence, $\theta_i = 0$ is a minimizer. Plugging this into our objective function,

$$\begin{aligned}
h_i(\theta_{-i}) = \min_{\theta_i} \Bigg\{ & \max\{v_i(\alpha(\theta),\theta_i),\, \lambda_i C(\alpha(\theta))\} - \left(\sum_{j\neq i} v_j(\alpha_{-i}(\theta_{-i}),\theta_j) - C(\alpha_{-i}(\theta_{-i}))\right) \\
& + \left(\sum_{j\neq i} v_j(\alpha(\theta),\theta_j) - C(\alpha(\theta))\right) \Bigg\} \\
= {} & \max\{v_i(\alpha_{-i}(\theta_{-i}),0),\, \lambda_i C(\alpha_{-i}(\theta_{-i}))\} - \left(\sum_{j\neq i} v_j(\alpha_{-i}(\theta_{-i}),\theta_j) - C(\alpha_{-i}(\theta_{-i}))\right) \\
& + \left(\sum_{j\neq i} v_j(\alpha_{-i}(\theta_{-i}),\theta_j) - C(\alpha_{-i}(\theta_{-i}))\right) \\
= {} & \lambda_i C(\alpha_{-i}(\theta_{-i}))
\end{aligned}$$

■

Theorem 1 tells us two things. The first is that the cost-sharing pivot mechanism indeed satisfies our three axioms. But there are several such mechanisms. The second is that, among these mechanisms, the cost-sharing pivot minimizes the budget-deficit. However, while minimizing a positive budget-deficit is desirable, minimizing a negative budget-deficit (i.e. maximizing a positive budget surplus) is not. In other words, our primary goal is not to maximize ex-post revenue, but rather to achieve or come close to achieving ex-post budget-balance.

Definition 8. A mechanism $f = (\alpha, \tau)$ is *ex-post budget-balanced (EPBB)* if

$$\sum_{i=1}^{n} \tau_i(\theta) - C(\alpha(\theta)) = 0.$$

This says that, after collecting transfers and paying implementation costs, there is no remaining surplus or deficit.

Hence, at this stage we should have two main concerns. While the cost-sharing pivot does raise the most revenue among all mechanisms that satisfy our axioms, we haven't said anything about how much revenue this is. It may be that the cost-sharing pivot raises *too little* revenue (in which case we would conclude there exists no satisfactory mechanism). On the other hand, the cost-sharing pivot may raise *too much* revenue. That is, there may exist another mechanism that satisfies the axioms and gets closer to EPBB than the cost-sharing pivot. The following theorems address both concerns.

Before we state them, let me point out that these theorems do *not* depend on our non-negativity assumption. Hence, let us define $\mathbb{E}_B$ to be the set of environments that replace Assumption 1 with the weaker Assumption 2.

Assumption 2. For each individual i, the set of admissible values

$$\Theta_i \in \{[a_i, b_i] : b_i > a_i\} \cup \{(-\infty, b_i] : b_i > -\infty\} \cup \{[a_i, \infty) : a_i < \infty\} \cup (-\infty, \infty).$$

Theorem 2. *Let θ_i be a sequence of i.i.d. random variables for which $E[\theta_i^2] < \infty$. Then given any environment $\mathcal{E} \in \mathbb{E}_B$, the probability of ex-post budget-balance in any cost-sharing pivot mechanism goes to one as n goes to infinity holding the per person cost $\bar{c} = c/n$ constant, i.e.*

$$P\left(\sum_{i=1}^{n} \tau_i(\theta) - C(\alpha(\theta)) = 0\right) \to 1 \quad as \quad n \to \infty.$$

Proof. See Appendix C. ■

Let us refer to an outcome as "undesirable" if the government must redistribute a surplus or a finance a deficit. Then Theorem 2 tells us that, for *any* beliefs about the underlying value distribution (for which θ_i are i.i.d. and $E[\theta_i^2] < \infty$), the probability of an undesirable outcome is arbitrarily small for large populations.

However, while this tells us about the asymptotic *frequency* of undesirable outcomes, it does not tell us about the asymptotic *magnitude* of the undesirability of such outcomes. That is, we may worry that while undesirable outcomes become increasingly unlikely, they also become increasingly severe. The following two theorems quell this concern.

Theorem 3. *Let (θ_i, z_i) be a sequence of i.i.d. random vectors for which $z_i \geq \underline{z} > 0$, $E[\theta_i] \neq \bar{c}$, and $E[\theta_i^2] < \infty$. Suppose*

$$\exists \alpha, \beta \in \mathbb{R}, \forall \theta_i \in \Theta_i,\ E\left[z_i \,|\, \theta_i\right] \leq \alpha|\theta_i| + \beta. \tag{1}$$

Then given any environment $\mathcal{E} \in \mathbb{E}_B$, the expected distance from ex-post budget balance in the cost-sharing pivot mechanism with cost shares $\lambda_i = \frac{z_i}{\sum_{j=1}^{n} z_j}$ goes to zero as n goes to infinity holding the per person cost $\bar{c} = c/n$ constant, i.e.

$$E\left[\left|\sum_{i=1}^{n} \tau_i(\theta) - C(\alpha(\theta))\right|\right] \to 0 \quad as \quad n \to \infty.$$

Proof. See Appendix C. ■

Theorem 4. *Let* (θ_i, z_i) *be a sequence of i.i.d. random vectors for which* $z_i \geq \underline{z} > 0$, $E[\theta_i] = \bar{c}$, *and* $E[\theta_i^2] < \infty$. *Suppose*

$$\exists \alpha, \beta \in \mathbb{R}, \forall \theta_i \in \Theta_i, \; E\left[z_i \,|\, \theta_i\right] \leq \alpha|\theta_i| + \beta.$$

Then given any environment $\mathcal{E} \in \mathbb{E}_B$, *the expected distance from ex-post budget balance* per person *in the cost-sharing pivot mechanism with cost shares* $\lambda_i = \frac{z_i}{\sum_{j=1}^n z_j}$ *goes to zero as* n *goes to infinity holding the per person cost* $\bar{c} = c/n$ *constant, i.e.*

$$\frac{1}{n} E\left[\left|\sum_{i=1}^{n} \tau_i(\theta) - C(\alpha(\theta))\right|\right] \to 0 \quad as \quad n \to \infty.$$

Proof. See Appendix C. ∎

Theorem 2 tells us that undesirable outcomes become increasingly unlikely, and Theorems 3 and 4 tell us that the undesirability of such outcomes also becomes increasingly small. Hence, we shall call any mechanism that satisfies all three asymptotically ex-post budget-balanced.

Definition: A mechanism f is *asymptotically ex-post budget-balanced (AEPBB)* if Theorems 2, 3, and 4 (appropriately altered) hold for f.

Let us take a moment to unpack these last two theorems. In Theorem 2, there is no mention of exogenous data z which may be used to inform the cost shares λ. This is because the probability of an undesirable outcome only depends on the values θ—not the cost shares.[16] However, in Theorems 3 and 4, we are interested in the magnitude of deviation from EPBB, and this *does* depend on the cost shares.

First, suppose equal cost shares, so that $\lambda_i = \frac{1}{n}$ and $\lambda_i c = \bar{c}$ is held fixed as we take n to infinity. Notice that (1) is immediately satisfied (in this case z_i is trivially some constant $\bar{z}$ for all i). Then Theorem 3 tells us that, for *any* belief for which θ_i are i.i.d., $E[\theta_i] \neq \bar{c}$, and $E[\theta_i^2] < \infty$, the expected distance from EPBB is arbitrarily small for large populations. On the other hand, in the knife-edge case that beliefs have $E[\theta_i] = \bar{c}$, Theorem 4 tells us that the expected distance from EPBB *per person* is arbitrarily small for large populations. In other words, if the population were to be equally rebated to disperse any surplus or equally taxed to fund any deficit, the expected per-person rebates/taxes would be arbitrarily small for large populations.

To gain some intuition, notice that if no players are pivotal[17], transfers are $\tau_i(\theta) = \lambda_i C(\alpha_{-i}(\theta_{-i})) = \lambda_i C(\alpha(\theta))$ for all i and so the cost-sharing pivot is EPBB. Thus,

[16] Technically, the probability that there is at least one pivotal player does not depend on the cost shares, which is an upper bound on the probability of an undesirable outcome.

[17] Player i is pivotal if $\alpha(\theta) \neq \alpha_{-i}(\theta_{-i})$.

undesirable outcomes only occur when at least one player is pivotal. Theorem 2 tells us that this probability goes to zero as n gets large. But what about i's expected payment conditional on being pivotal, or more particularly the expected *distance* between i's payment and her equal share $\bar{c}$ conditional on being pivotal? It turns out this conditional expected distance diverges with n, but Theorems 3 and 4 tell us that the probability of at least one pivotal player goes to zero sufficiently quickly so as to out-pace both the increasing population size and diverging expected distance between a pivotal player's payment and her equal share.

Now, let us consider cost shares that may depend on (non-trivial) exogenous data. Suppose that we map some available data into indices $z = (z_1, \ldots, z_n)$ such that i's share of the cost is precisely her relative share of z, i.e. $\lambda_i = \frac{z_i}{\sum_{j=1}^n z_j}$. Assume for simplicity that we are in an environment with non-negative values.

Suppose player i is pivotal. Then her payment is $c - \sum_{j \neq i} \theta_j$, which is not in general equal to her share of the cost $\lambda_i c$. The probability that a player is pivotal is low, so we can imagine most of i's opponents j are not pivotal, in which case they will contribute their fair share $\lambda_j c$. Hence, if i contributes less (more) than her share, we may run a deficit (surplus). Moreover, if we happen to believe that higher values θ_i correlate with higher z_i, then we would expect that pivotal players (who are likely to have large θ_i) have large z_i, and hence large λ_i. If λ_i is very large, i's share will be near the full cost c, and i's non-pivotal opponents will contribute next to nothing. Since i's pivotal payment $c - \sum_{j \neq i} \theta_j$ is by definition strictly less than c (and has nothing to do with her share λ_i), this may result in a considerable deficit. Additionally, a larger n implies a larger expected θ_i conditional on i pivotal, leading to a larger expected λ_i. A case of such extreme beliefs may reverse the positive asymptotic results with equal cost shares. Bounding this type of correlation is precisely the purpose of (1), which says that the expected index z_i conditional on i's value θ_i is bounded by *some* affine function of $|\theta_i|$. It turns out this is sufficient to recover the positive asymptotic results with equal cost shares.

To conclude this section, let us return to our two original concerns: that the cost-sharing pivot does not raise enough revenue, and that the cost-sharing pivot raises too much revenue (i.e. there may exist another mechanism that satisfies the axioms and gets closer to EPBB). The first concern is clearly resolved by AEPBB and Theorem 1. The second is resolved by AEPBB, Theorem 1, and Theorem 5.

Theorem 5. *Given an environment $\mathcal{E} \in \mathbb{E}_{B_+}$ for which the set of admissible values is the same across individuals $\Theta_i = \Theta_j$ for all $i, j \in I$, there is no strategy-proof, efficient, and extortion-free mechanism that gets closer to EPBB for every θ than the cost-sharing pivot mechanism.*

Proof. Given any environment $\mathcal{E} \in \mathbb{E}_{B_+}$, a mechanism $f = (\alpha, \tau)$ is strategy-proof (SP) and efficient (E) if and only if it is a VCG mechanism by Holmström (1979), pinning down the transfers up to a constant in i's report. From Theorem 1, we know that SP, E, and no-extortion (NE) place an upper bound on transfers. That is, transfers must satisfy

$$\tau_i(\theta) = \begin{cases} 0 - r_i^0(\theta_{-i}) & \text{if } \alpha(\theta) = 0 \\ c - \sum_{j \neq i} \theta_j - r_i^0(\theta_{-i}) & \text{if } \alpha_{-i}(\theta_{-i}) = 0 \text{ and } \alpha(\theta) = 1 \\ \lambda_i c - r_i^1(\theta_{-i}) & \text{if } \alpha_{-i}(\theta_{-i}) = 1 \text{ and } \alpha(\theta) = 1 \end{cases}$$

where $r_i^0(\theta_{-i})$, $r_i^1(\theta_{-i}) \geq 0$ for all $\theta_{-i} \in \Theta_{-i}$. The cost-sharing pivot mechanism sets transfers equal to their upper bound, i.e. $r_i^0(\theta_{-i}) = r_i^1(\theta_{-i}) = 0$ for all θ_{-i}.

Suppose the cost-sharing pivot runs a budget deficit for some $\tilde{\theta}$. No other SP, E, and NE mechanism can come closer to EPBB by Theorem 1.

Suppose the cost-sharing pivot runs a budget surplus for some $\hat{\theta}$. This implies that the good is provided and at least one player is pivotal. We seek to find at least one player whose transfer we can reduce by some rebate $r_i^k(\hat{\theta}_{-i}) > 0$ that brings us strictly closer to EPBB for $\hat{\theta}$, such that the cost-sharing pivot is not strictly closer to EPBB for *any* $(\theta_i, \hat{\theta}_{-i})$.

Suppose we choose a pivotal player j to rebate $r_j^0(\hat{\theta}_{-j}) > 0$. If $\theta_j = 0$, the good would not be provided, and we would run a strict budget deficit (since j is paying less than her fair share [zero] and all others are paying their fair share [zero], which is the max we can extract from them), while the cost-sharing pivot is EPBB for $(0, \hat{\theta}_{-j})$.

Suppose we choose a non-pivotal player i to rebate $r_i^1(\hat{\theta}_{-i}) > 0$. If $\theta_i \geq \max_{j \neq i} \hat{\theta}_j$ (which is admissible by assumption), all players would be non-pivotal, since

$$\forall k, \sum_{j \neq k} \theta_j \geq c \iff \sum_{j=1}^{n} \theta_j - \max_k \theta_k \geq c \iff \sum_{j \neq i} \theta_j \geq c,$$

which is true since i is non-pivotal. Thus, we would run a strict budget deficit (since i is paying less than her fair share of c and all others are paying their fair share, which is the max we can extract from them), while the cost-sharing pivot is EPBB for $(\theta_i, \hat{\theta}_{-i})$.

■

The cost-sharing pivot is not the only mechanism that satisfies strategy-proofness, efficiency, no-extortion, and AEPBB. We can construct mechanisms that deviate from the cost-sharing pivot by increasingly small amounts, e.g. by providing rebates that vanish in magnitude and/or probability with n. For example, consider a mechanism providing a one dollar rebate to individual i if all of i's opponents report zero. No matter the probability with which we believe the zero type is drawn, the probability that any i receives a rebate goes to zero as n goes to infinity.[18] However, Theorem 5 tells us that there is no mechanism that brings us closer to EPBB for all θ than the cost-sharing pivot mechanism. In other words, there is no mechanism that always outperforms it.

Thus, we can justify the choice of the cost-sharing pivot in three steps. Denote the set of all mechanisms that are strategy-proof, efficient, extortion-free, and AEPBB by F. First, the cost-sharing pivot is simple and intuitive. Any alternative mechanism in F inevitably adds additional layers of complexity. Second, the cost-sharing pivot is undominated, in that there is no mechanism in F that outperforms it for all θ. Third, the cost-sharing pivot minimizes the ex-post budget-deficit among all mechanisms in F. Thus, if we feel more strongly about deficits than surpluses, the cost sharing pivot is a natural choice.

5 Conclusion

We have shown that there exists a satisfactory (SP, E, NE, and AEPBB) mechanism for the provision of a costly public good in environments where individuals can be assumed to have non-negative values for the good and the government has the power to tax.[19] Moreover, one such mechanism, the cost-sharing pivot, is simple, detail-free, undominated, and uniquely minimizes any potential budget deficit.

One way to understand the cost-sharing pivot is as a slight adjustment to the so-called "naive cost-sharing mechanism", in which the good is provided if and only if it is efficient to do so and the players split the cost according to fixed shares λ if and only if it is provided. This mechanism gives highly perverse incentives, regardless of the population size: if i's value θ_i is greater than her cost share $\lambda_i c$, it is weakly dominant to report an *infinitely* large value; if her value is less than her cost share, it is weakly dominant to report *zero*.

The cost-sharing pivot mechanism modifies these transfers by requiring special payments from pivotal players. Namely, if i's report sways the decision from not-providing to providing, she pays the minimum value required to sway the decision $c - \sum_{j \neq i} \theta_j$ instead of her cost share $\lambda_i c$. The mechanisms are otherwise identical. But notice that the probability that a player actually makes this special payment goes to zero with n. Hence, this "threat" removes infinitely perverse incentives, even though the probability that it is actually carried out goes to zero. The cost-sharing pivot can thus be seen as a tweak to the naive cost-sharing mechanism that

[18] Let $p \equiv P(\theta_i = 0) < 1$. Then $P(\exists i \in I, \forall j \neq i, \theta_j = 0) \leq \sum_{i=1}^{n} P(\forall j \neq i, \theta_j = 0) = np^{n-1} \to 0$ as $n \to \infty$.

[19] We have also identified the serious, and possibly unexpected, challenges of costly public goods provision when negative values are admissible.

repairs its gross manipulability while sacrificing little of its otherwise exceptional properties.[20]

A No-Arbitrage and EPBS

Definition 9. A mechanism $f = (\alpha, \tau)$ satisfies *no-arbitrage* if for all i,

$$\forall \theta : \alpha(\theta) = 0 \text{ and } \theta_i \geq 0, \quad \tau_i(\theta) \leq 0.$$

Definition 10. A player i is *potentially pivotal(+) against θ_{-i}* if

$$\alpha_{-i}(\theta_{-i}) = 0, \quad \exists \theta_i \in \Theta_i \ : \ \alpha(\theta) = 0, \quad \text{and} \quad \exists \theta_i \in \Theta_i \ : \ \alpha(\theta) = 1.$$

Lemma 1. *Given an environment $\mathcal{E} \in \mathbb{E}_B$, the unique ex-post revenue-maximizing mechanism f among all SP, E, and NA mechanisms is a VCG mechanism with pivot transfers for potentially pivotal(+) players. That is, if player i is potentially pivotal(+) against θ_{-i}, then*

$$\tau_i(\theta) = \sum_{j \neq i} v_j(\alpha_{-i}(\theta_{-i}), \theta_j) - C(\alpha_{-i}(\theta_{-i})) - \left(\sum_{j \neq i} v_j(\alpha(\theta), \theta_j) - C(\alpha(\theta)) \right).$$

Proof. Given any environment $\mathcal{E} \in \mathbb{E}_B$, a mechanism f is strategy-proof and efficient if and only if it is a VCG mechanism by Holmström (1979). Denote VCG transfers by

$$\tau_i(\theta) = \sum_{j \neq i} v_j(\alpha_{-i}(\theta_{-i}), \theta_j) - C(\alpha_{-i}(\theta_{-i})) - \left(\sum_{j \neq i} v_j(\alpha(\theta), \theta_j) - C(\alpha(\theta)) \right) + h_i(\theta_{-i})$$

for any function $h_i : \Theta_{-i} \to \mathbb{R}$. Suppose i is potentially pivotal(+) against θ_{-i}. Then NA requires

$$\tau_i(\theta) = \sum_{j \neq i} v_j(0, \theta_j) - C(0) - \left(\sum_{j \neq i} v_j(0, \theta_j) - C(0) \right) + h_i(\theta_{-i}) \leq 0 \quad \forall \theta_i \geq 0 \text{ s.t. } \alpha(\theta) = 0$$

$$\iff h_i(\theta_{-i}) \leq 0 \quad \forall \theta_i$$

where the last line follows since h_i cannot depend on θ_i directly. Hence, the unique ex-post revenue maximizing mechanism among all SP, E, and NA mechanisms sets $h_i(\theta_{-i}) = 0$ whenever i is potentially pivotal(+) against θ_{-i}.

Notice that the $\theta_i \geq 0$ requirement of NA turns out not to matter. ■

Lemma 2. *Given any environment $\mathcal{E} \in \mathbb{E}_B$ such that for some $\delta > 0$ and $\varepsilon \in (0, \frac{c}{n}]$, $[\varepsilon - \delta, \frac{c-\varepsilon}{n-1}] \subseteq \Theta_i$ for all i, there exists a θ such that each player i is potentially pivotal(+)*

[20] Without the glaring incentive problem (which would destroy these other properties), the naive cost-sharing mechanism is E, NE, and EPBB. The cost-sharing pivot is SP, E, NE, and AEPBB.

and the good is provided. Moreover, in such a case, a VCG mechanism with pivot transfers for potentially pivotal(+) players will result in a strict budget deficit,

$$\sum_{i \in I} \tau_i(\theta) < c.$$

Proof. Part I: We would like to show that, for any n and c, there exists a θ such that each player i is potentially pivotal(+) against θ_{-i}, $\sum_{j \neq i} \theta_j < c$, and the good is provided, $\sum_{i \in I} \theta_i > c$.

Suppose that $\theta_i = \frac{c-\varepsilon}{n-1}$ for all i with $\varepsilon \in (0, \frac{c}{n}]$. Then

$$\sum_{j \neq i} \theta_j = c - \varepsilon < c \quad \text{and} \quad \sum_{i \in I} \theta_i = \frac{n}{n-1}(c - \varepsilon) \geq c,$$

since

$$\frac{n}{n-1}(c-\varepsilon) \geq c \iff (c-\varepsilon) \geq \frac{n-1}{n}c \iff c - \frac{n-1}{n}c \geq \varepsilon \iff \frac{c}{n} \geq \varepsilon.$$

On the other hand, if any $\theta_i = \varepsilon - \delta$, $\sum_{i \in I} \theta_i = c - \delta < c$. Hence, each i is potentially pivotal(+) against θ_{-i}.

Notice in this case, each player pays ε for a total revenue of $n\varepsilon$.

A simple example is $\Theta_i = [0, \infty)$, $n = 3$, $c = 8$, and $\theta = (3, 3, 3)$. □

Part II: Suppose θ is such that each player i is potentially pivotal(+) against θ_{-i} and the public good is provided. The total revenue raised by a VCG mechanism with pivot transfers for potentially pivotal(+) players is then

$$\begin{aligned}
\sum_{i \in I} \tau_i(\theta) &= \sum_{i \in I} \left(c - \sum_{j \neq i} \theta_j \right) \\
&= Nc - \sum_{i \in I} \sum_{j \neq i} \theta_j \\
&= Nc - (N-1) \sum_{i \in I} \theta_i \\
&= (N-1)c - (N-1) \sum_{i \in I} \theta_i + c \\
&= (N-1) \left(c - \sum_{i \in I} \theta_i \right) + c \\
&< c
\end{aligned}$$

where the last line follows since $c - \sum_{i \in I} \theta_i < 0$ (the good is provided). ■

Proposition 2. *Given an environment $\mathcal{E} \in \mathbb{E}_B$ such that for some $\delta > 0$ and $\varepsilon \in (0, \frac{c}{n}]$, $[\varepsilon - \delta, \frac{c-\varepsilon}{n-1}] \subseteq \Theta_i$ for all i, there does not exist any mechanism $f : \Theta \to X$ that satisfies SP, E, NA, and EPBS.*

Proof. The proof follows directly from Lemmas 1 and 2. By Lemma 1, the unique ex-post revenue-maximizing mechanism f among all SP, E, and NA mechanisms is a VCG mechanism with pivot transfers for potentially pivotal(+) players and, by Lemma 2, when everyone is potentially pivotal(+) and the public good is provided (which is possible for any n and c), such a mechanism runs a strict ex-post budget deficit, violating EPBS. ■

Corollary 1. *Given an environment $\mathcal{E} \in \mathbb{E}_B$ such that for some $\delta > 0$ and $\varepsilon \in (0, \frac{c}{n}]$, $[\varepsilon - \delta, \frac{c-\varepsilon}{n-1}] \subseteq \Theta_i$ for all i, there does not exist any mechanism $f : \Theta \to X$ that satisfies SP, E, EPIR, and EPBS.*

Proof. EPIR implies NA, i.e.

$$\forall\theta,\ \tau_i(\theta) \leq v_i(\alpha(\theta), \theta_i) \implies \forall\theta : \alpha(\theta) = 0 \text{ and } \theta_i \geq 0,\ \tau_i(\theta) \leq 0.$$

Hence, the result follows immediately from Proposition 2. ■

B Challenges of Negative Values

Definition 11. A mechanism $f = (\alpha, \tau)$ satisfies *EPIR-no-extortion (EPIR-NE)* if

$$\tau_i(\theta) \leq \max\{v_i(\alpha(\theta), \theta_i),\ \lambda_i C(\alpha(\theta))\} \quad \forall\theta.$$

Definition 12. A mechanism $f = (\alpha, \tau)$ satisfies *UP-no-extortion (UP-NE)* if

$$\tau_i(\theta) \leq \max\{v_i(\alpha(\theta), \theta_i) - v_i(\alpha_{-i}(\theta_{-i}), \theta_i),\ \lambda_i C(\alpha(\theta))\} \quad \forall\theta.$$

Definition 13. A mechanism $f = (\alpha, \tau)$ is a *plain-vanilla VCG* if the decision rule is efficient and the transfer rule satisfies

$$\tau_i(\theta) = -\left(\sum_{j \neq i} v_j(\alpha(\theta), \theta_j) - C(\alpha(\theta))\right)$$

The following proposition identifies the revenue-maximizing mechanism under SP, E, and either EPIR-NE or UP-NE in binary decision environments with no lower bound on admissible values. Recall that SP and EPIR-NE are equivalent to SP and UP-NE in environments with non-negative values. However, this is not true with negative values, so we must analyze both conditions separately.

Proposition 3. *Given any binary decision environment with no lower bound on admissible values, the revenue-maximizing mechanism among all strategy-proof, efficient, and EPIR-NE mechanisms is the plain-vanilla VCG, and the revenue-maximizing mechanism among all strategy-proof, efficient, and UP-NE mechanisms is the pivot mechanism.*

The proof uses the same techniques as Theorem 1 and is omitted. The following proposition tells us that these mechanisms are *not* approximately EPBB and, in fact, diverge from EPBB with n.

Proposition 4. *Let θ_i be a sequence of i.i.d. random variables for which $E[\theta_i] > \bar{c}$ and $E[\theta_i^2] < \infty$. Then given any environment $\mathcal{E} \in \mathbb{E}_B$, the probability of ex-post budget-surplus in a pivot mechanism goes to zero as n goes to infinity holding the per person cost $\bar{c} = c/n > 0$ constant, i.e.*

$$P\left(\sum_{i=1}^{n} \tau_i(\theta) - C(\alpha(\theta)) \geq 0\right) \to 0 \quad as \quad n \to \infty.$$

Moreover, the expected distance from EPBB goes to infinity as n goes to infinity, i.e.

$$E\left[\left|\sum_{i=1}^{n} \tau_i(\theta) - C(\alpha(\theta))\right|\right] \to \infty \quad as \quad n \to \infty.$$

Likewise for a plain-vanilla VCG.

Proof. It will suffice to show these results for the pivot mechanism, since the plain-vanilla VCG raises strictly less revenue than the pivot.

For the first part,

$$\begin{aligned}
&P\left(\sum_{i=1}^{n} \tau_i(\theta) - C(\alpha(\theta)) \geq 0\right) \\
&\leq P\left(\sum_{i=1}^{n} \theta_i \leq (n\bar{c})\right) + P\left(\exists \text{ a pivotal player}\right) \\
&\to 0 \quad \text{as} \quad n \to \infty
\end{aligned}$$

by the LLN and the same arguments as in the proof of Theorem 2. If Y is a random variable on the probability space $(\Omega, \mathcal{A}, P)$ and $B \in \mathcal{A}$ is an event, then let $E\left[Y, B\right] = \int_B Y \ \mathrm{d}P$. Then for the second part,

$$\begin{aligned}
&E\left[\left|\sum_{i=1}^{n} \tau_i(\theta) - C(\alpha(\theta))\right|\right] \\
&\geq \sum_{i=1}^{n} E\left[\lambda_i C(\alpha(\theta)) - \tau_i(\theta)\right] \\
&\geq \sum_{i=1}^{n} E\left[\lambda_i(n\bar{c}), \ \sum_{j\neq i} \theta_j \geq (n\bar{c}) \ \text{ and } \ \theta_i + \sum_{j\neq i} \theta_j \geq (n\bar{c})\right] \\
&= P\left(\sum_{j\neq i} \theta_j \geq (n\bar{c}) \ \text{ and } \ \theta_i + \sum_{j\neq i} \theta_j \geq (n\bar{c})\right) \sum_{i=1}^{n} \lambda_i(n\bar{c}) \\
&= P\left(\theta_i + \sum_{j\neq i} \theta_j \geq (n\bar{c}) \,|\, \sum_{j\neq i} \theta_j \geq (n\bar{c})\right) \cdot P\left(\sum_{j\neq i} \theta_j \geq (n\bar{c})\right) \cdot \bar{c} \cdot n \\
&\geq P\left(\theta_i \geq 0\right) \cdot P\left(\frac{1}{n-1} \sum_{j\neq i} \theta_j \geq \frac{n}{n-1}\bar{c}\right) \cdot \bar{c} \cdot n \\
&\to \infty \quad \text{as} \quad n \to \infty
\end{aligned}$$

since the left term is a constant in n and the middle term converges to one by the LLN. ■

C Asymptotic Proofs

Lemma 3. *Let $\bar{c} = c/n$ and $\bar{\theta}_i = \theta_i - \bar{c}$. In a cost-sharing pivot mechanism,*

$$\tau_i(\theta) - \lambda_i C(\alpha(\theta)) = \begin{cases} 0 & \textit{if } \alpha(\theta) = 0 \textit{ and } i \textit{ is not pivotal}(-) \\ \sum_{j \neq i} \bar{\theta}_j + (\bar{c} - c_i) & \textit{if } \alpha(\theta) = 0 \textit{ and } i \textit{ is pivotal}(-) \\ 0 & \textit{if } \alpha(\theta) = 1 \textit{ and } i \textit{ is not pivotal}(+) \\ -\sum_{j \neq i} \bar{\theta}_j + (\bar{c} - c_i) & \textit{if } \alpha(\theta) = 1 \textit{ and } i \textit{ is pivotal}(+) \end{cases}.$$

and

$$i \textit{ is pivotal} \implies \left|\bar{\theta}_i\right| \geq \left|\sum_{j \neq i} \bar{\theta}_j\right| \textit{ or } \left|\sum_{j \neq i} \bar{\theta}_j\right| \leq c_i.$$

Proof. In a cost-sharing pivot mechanism,

$$\tau_i(\theta) = \begin{cases} 0 & \text{if } \alpha(\theta) = 0 \text{ and } i \text{ is not pivotal}(-) \\ \sum_{j \neq i} \theta_j - c + c_i & \text{if } \alpha(\theta) = 0 \text{ and } i \text{ is pivotal}(-) \\ c_i & \text{if } \alpha(\theta) = 1 \text{ and } i \text{ is not pivotal}(+) \\ -(\sum_{j \neq i} \theta_j - c) & \text{if } \alpha(\theta) = 1 \text{ and } i \text{ is pivotal}(+) \end{cases}$$

and hence

$$\tau_i(\theta) - \lambda_i C(\alpha(\theta)) = \begin{cases} 0 & \text{if } \alpha(\theta) = 0 \text{ and } i \text{ is not pivotal}(-) \\ \sum_{j \neq i}(\theta_j - \bar{c}) - \bar{c} + c_i & \text{if } \alpha(\theta) = 0 \text{ and } i \text{ is pivotal}(-) \\ 0 & \text{if } \alpha(\theta) = 1 \text{ and } i \text{ is not pivotal}(+) \\ -\left(\sum_{j \neq i}(\theta_j - \bar{c}) - \bar{c} + c_i\right) & \text{if } \alpha(\theta) = 1 \text{ and } i \text{ is pivotal}(+) \end{cases}.$$

Moreover,

$$\begin{aligned} i \text{ is pivotal}(+) &\iff \sum_{j \neq i} \theta_j < c \text{ and } \sum_{i \in I} \theta_i \geq c \\ &\iff \sum_{j \neq i} \bar{\theta}_j < \bar{c} \text{ and } \sum_{i \in I} \bar{\theta}_i \geq 0 \\ &\iff \left(\sum_{j \neq i} \bar{\theta}_j < 0 \text{ and } \bar{\theta}_i \geq -\sum_{j \neq i} \bar{\theta}_j\right) \text{ or } \left(\sum_{j \neq i} \bar{\theta}_j \in [0, \bar{c}] \text{ and } \bar{\theta}_i \geq -\sum_{j \neq i} \bar{\theta}_j\right) \\ &\implies \left|\bar{\theta}_i\right| \geq \left|\sum_{j \neq i} \bar{\theta}_j\right| \text{ or } \left|\sum_{j \neq i} \bar{\theta}_j\right| \leq \bar{c} \end{aligned}$$

and pivotal(−) if and only if

$$\begin{aligned} i \text{ is pivotal}(-) &\iff \sum_{j\neq i}\theta_j \geq c \ \text{ and } \ \sum_{i\in I}\theta_i < c \\ &\iff \sum_{j\neq i}\bar{\theta}_j \geq \bar{c} \ \text{ and } \ \sum_{i\in I}\bar{\theta}_i < 0 \\ &\iff \sum_{j\neq i}\bar{\theta}_j \geq \bar{c} \ \text{ and } \ \bar{\theta}_i < -\sum_{j\neq i}\bar{\theta}_j \\ &\implies \left|\bar{\theta}_i\right| \geq \left|\sum_{j\neq i}\bar{\theta}_j\right|. \end{aligned}$$

Thus,

$$i \text{ is pivotal} \implies \left|\bar{\theta}_i\right| \geq \left|\sum_{j\neq i}\bar{\theta}_j\right| \ \text{ or } \ \left|\sum_{j\neq i}\bar{\theta}_j\right| \leq \bar{c}.$$

■

Lemma 4. *Let θ_i be a sequence of i.i.d. random variables and $c \in \mathbb{R}$. If $E[\theta_i^2] < \infty$,*

$$\frac{c}{\left|\sum_{j\in I}\theta_j\right|} \xrightarrow{p} 0.$$

Proof. If $E[\theta_i] \neq 0$, $\left|\sum_{j\in I}\theta_j\right|/n \xrightarrow{a.s.} |E[\theta_i]|$ by the strong law of large numbers (see. Loeve p. 251), and so

$$\frac{c/n}{\left|\sum_{j\in I}\theta_j\right|/n} \xrightarrow{a.s.} 0.$$

If $E[\theta_i] = 0$,

$$\frac{c/\sigma\sqrt{n}}{\left|\sum_{j\in I}\theta_j\right|/\sigma\sqrt{n}} \xrightarrow{p} 0$$

by Rob (1982, pp. 212-213, Proof of Lemma 2), where $\sigma^2 = E[\theta_i^2]$. ■

Theorem 2. *Let θ_i be a sequence of i.i.d. random variables for which $E[\theta_i^2] < \infty$. Then given any environment $\mathcal{E} \in \mathbb{E}_B$, the probability of ex-post budget-balance in any cost-sharing pivot mechanism goes to one as n goes to infinity holding the per person cost $\bar{c} = c/n$ constant, i.e.*

$$P\left(\sum_{i=1}^{n}\tau_i(\theta) - C(\alpha(\theta)) = 0\right) \to 1 \quad as \quad n \to \infty.$$

Proof. It suffices to show that

$$P\left(\sum_{i=1}^{n}\tau_i(\theta) - C(\alpha(\theta)) \neq 0\right) \to 0 \quad \text{as} \quad n \to \infty.$$

Let $\bar{c} = c/n$, $\bar{\theta}_i = \theta_i - \bar{c}$, and $\bar{\theta}_n^* = \max_i |\bar{\theta}_i|$. Note $E[\bar{\theta}_i^2] < \infty$. Then,

$$
\begin{aligned}
&P\left(\sum_{i=1}^{n} \tau_i(\theta) - C(\alpha(\theta)) \neq 0\right) \\
&\leq P\left(\exists \text{ a pivotal player}\right) \\
&\leq P\left(|\bar{\theta}_i| \geq \left|\sum_{j\neq i} \bar{\theta}_j\right| \text{ or } \left|\sum_{j\neq i} \bar{\theta}_j\right| \leq \bar{c} \text{ for some } i\right) \\
&\leq P\left(|\bar{\theta}_i| \geq \left|\sum_{j\neq i} \bar{\theta}_j\right| \text{ for some } i\right) + P\left(\left|\sum_{j\neq i} \bar{\theta}_j\right| \leq \bar{c} \text{ for some } i\right) \\
&= P\left(|\bar{\theta}_i| + |\bar{\theta}_i| \geq \left|\sum_{j\neq i} \bar{\theta}_j + |\bar{\theta}_i|\right| \text{ for some } i\right) + P\left(\left|\sum_{j\neq i} \bar{\theta}_j\right| + |\bar{\theta}_i| \leq \bar{c} + |\bar{\theta}_i| \text{ for some } i\right) \\
&\leq P\left(2|\bar{\theta}_i| \geq \left|\sum_{j\in I} \bar{\theta}_j\right| \text{ for some } i\right) + P\left(|\bar{\theta}_i| + \bar{c} \geq \left|\sum_{j\in I} \bar{\theta}_j\right| \text{ for some } i\right) \\
&= P\left(\frac{\bar{\theta}_n^*}{\left|\sum_{j\in I} \bar{\theta}_j\right|} \geq \frac{1}{2}\right) + P\left(\frac{\bar{\theta}_n^*}{\left|\sum_{j\in I} \bar{\theta}_j\right|} + \frac{\bar{c}}{\left|\sum_{j\in I} \bar{\theta}_j\right|} \geq 1\right) \\
&\to 0 \quad \text{as} \quad n \to \infty
\end{aligned}
$$

by Lemma 3, Rob (1982, p. 211-212, Lemma 1 and 2), and Lemma 4. ■

If Y is a random variable on the probability space $(\Omega, \mathcal{A}, P)$ and $B \in \mathcal{A}$ is an event, then let $E[Y, B] = \int_B Y \, \mathrm{d}P$.

Lemma 5. *Let (θ_i, w_i) be a sequence of i.i.d. draws from some distribution F with $w_i \geq \underline{w} > 0$ and let*

$$c_i \equiv \frac{w_i}{\sum_{j=1}^n w_j} c = \frac{w_i}{\frac{1}{n}\sum_{j=1}^n w_j} \bar{c}.$$

where $\bar{c} = c/n$. If

$$\exists \alpha, \beta \in \mathbb{R}, \forall \theta_i \in \Theta_i, \ E[w_i \mid \theta_i] \leq \alpha |\theta_i| + \beta,$$

then

$$E\left[c_i, \ |\theta_i| \geq \left|\sum_{j\neq i} \theta_j\right|\right] \leq \frac{\alpha \bar{c}}{\underline{w}} E\left[|\theta_i|, \ |\theta_i| \geq \left|\sum_{j\neq i} \theta_j\right|\right] + \frac{\beta \bar{c}}{\underline{w}} P\left(|\theta_i| \geq \left|\sum_{j\neq i} \theta_j\right|\right).$$

Proof. By assumption, there exists some $\alpha, \beta \in \mathbb{R}$ such that

$$E[w_i \mid \theta_i] = \int_{w_i \geq \underline{w}} w_i \, \mathrm{d}F(w_i \mid \theta_i) \leq \alpha |\theta_i| + \beta.$$

Hence,

$$
\begin{aligned}
E\left[w_i,\ |\theta_i| \geq K\right] &= \int_{|\theta_i|\geq K} \int_{w_i \geq \underline{w}} w_i \ \mathrm{d}F(w_i \,|\, \theta_i)\mathrm{d}F(\theta_i) \\
&\leq \int_{|\theta_i|\geq K} \alpha|\theta_i| + \beta \ \mathrm{d}F(\theta_i) \\
&= \alpha \int_{|\theta_i|\geq K} |\theta_i| \ \mathrm{d}F(\theta_i) + \beta \int_{|\theta_i|\geq K} \mathrm{d}F(\theta_i) \\
&= \alpha E\left[|\theta_i|,\ |\theta_i| \geq K\right] + \beta P(|\theta_i| \geq K)
\end{aligned}
$$

and

$$
\begin{aligned}
E\left[c_i,\ |\theta_i| \geq \left|\sum_{j\neq i} \theta_j\right|\right] &= E\left[\frac{w_i}{\frac{1}{n}\sum_{j=1}^n w_j}\bar{c},\ |\theta_i| \geq \left|\sum_{j\neq i}\theta_j\right|\right] \\
&\leq \frac{\bar{c}}{\underline{w}} E\left[w_i,\ |\theta_i| \geq \left|\sum_{j\neq i}\theta_j\right|\right] \\
&\leq \frac{\alpha\bar{c}}{\underline{w}} E\left[|\theta_i|,\ |\theta_i| \geq \left|\sum_{j\neq i}\theta_j\right|\right] + \frac{\beta\bar{c}}{\underline{w}} P\left(|\theta_i| \geq \left|\sum_{j\neq i}\theta_j\right|\right)
\end{aligned}
$$

■

Lemma 6. *If* $E[\theta_i] \neq 0$*,*

$$
nP\left(\left|\sum_{j\neq i}\theta_j\right| \leq |\theta_i|\right) \to 0 \quad as \quad n \to \infty.
$$

Proof. First, it will be useful to establish the following fact. By Markov's Inequality, if X is a non-negative random variable, then for any constant $c > 0$,

$$
P\left(X \geq c\right) \leq \frac{E[X]}{c}.
$$

Let $\theta_i^2 \equiv X$ and $\varepsilon^2 n^2 = c$, then

$$
\begin{aligned}
& P\left(\theta_i^2 \geq \varepsilon^2 n^2\right) \leq \frac{E[\theta_i^2]}{\varepsilon^2 n^2} \\
\iff & nP\left(\frac{|\theta_i|}{n} \geq \varepsilon\right) \leq \frac{E[\theta_i^2]}{\varepsilon^2 n} \to 0 \quad \text{as} \quad n \to \infty
\end{aligned} \tag{2}
$$

Now we may complete the proof. Let $S_{n-1} \equiv \sum_{j \neq i} \theta_j$. Suppose $E[\theta_i] > 0$, then

$$
\begin{aligned}
& nP\left(\left|\sum_{j \neq i} \theta_j\right| \leq |\theta_i|\right) \\
& = nP\left(E[S_{n-1}] - |\theta_i| \leq E[S_{n-1}] - |S_{n-1}|\right) \\
& \leq nP\left(E[S_{n-1}] - |\theta_i| \leq E[S_{n-1}] - S_{n-1}\right) \\
& \leq nP\left(E[S_{n-1}] - |\theta_i| \leq |E[S_{n-1}] - S_{n-1}|\right) \\
& = nP\left(E[\theta_i] - \frac{|\theta_i|}{n-1} \leq \frac{|S_{n-1} - E[S_{n-1}]|}{n-1}\right) \\
& = nP\left(\frac{|S_{n-1} - E[S_{n-1}]|}{n-1} + \frac{|\theta_i|}{n-1} \geq E[\theta_i]\right) \\
& = (n-1)P\left(\frac{|S_{n-1} - E[S_{n-1}]|}{n-1} + \frac{|\theta_i|}{n-1} \geq E[\theta_i]\right) + P\left(\frac{|S_{n-1} - E[S_{n-1}]|}{n-1} + \frac{|\theta_i|}{n-1} \geq E[\theta_i]\right) \\
& \to 0 \quad \text{as} \quad n \to \infty
\end{aligned}
$$

where the last line follows by Rob (1982, p. 215, Proof of Claim 2) and (2). If $E[\theta_i] < 0$,

$$
\begin{aligned}
& nP\left(\left|\sum_{j \neq i} \theta_j\right| \leq |\theta_i|\right) \\
& = nP\left(-E[S_{n-1}] - |\theta_i| \leq -E[S_{n-1}] - |S_{n-1}|\right) \\
& \leq nP\left(-E[S_{n-1}] - |\theta_i| \leq -E[S_{n-1}] + S_{n-1}\right) \\
& \leq nP\left(-E[\theta_i] - \frac{|\theta_i|}{n-1} \leq \frac{|S_{n-1} - E[S_{n-1}]|}{n-1}\right) \\
& = nP\left(\frac{|S_{n-1} - E[S_{n-1}]|}{n-1} + \frac{|\theta_i|}{n-1} \geq -E[\theta_i]\right) \\
& \to 0 \quad \text{as} \quad n \to \infty
\end{aligned}
$$

as before. ■

Theorem 3. *Let (θ_i, z_i) be a sequence of i.i.d. random vectors for which $z_i \geq \underline{z} > 0$, $E[\theta_i] \neq \bar{c}$, and $E[\theta_i^2] < \infty$. Suppose*

$$\exists \alpha, \beta \in \mathbb{R}, \forall \theta_i \in \Theta_i,\ E\left[z_i \mid \theta_i\right] \leq \alpha|\theta_i| + \beta. \tag{1}$$

Then given any environment $\mathcal{E} \in \mathbb{E}_B$, the expected distance from ex-post budget balance in the cost-sharing pivot mechanism with cost shares $\lambda_i = \frac{z_i}{\sum_{j=1}^n z_j}$ goes to zero as n goes to infinity holding the per person cost $\bar{c} = c/n$ constant, i.e.

$$E\left[\left|\sum_{i=1}^n \tau_i(\theta) - C(\alpha(\theta))\right|\right] \to 0 \quad \textit{as} \quad n \to \infty.$$

Proof. Let

$$c_i \equiv \frac{w_i}{\sum_{j=1}^n w_j} c = \frac{w_i}{\frac{1}{n}\sum_{j=1}^n w_j} \bar{c}.$$

where $\bar{c} = c/n$. Let $\bar{\theta}_i = \theta_i - \bar{c}$. Note $E[\bar{\theta}_i^2] < \infty$ and $E[c_i] = \bar{c}$. Then,

$$
\begin{aligned}
& E\left[\left|\sum_{i=1}^{n} \tau_i(\theta) - C(\alpha(\theta))\right|\right] \\
& \leq \sum_{i=1}^{n} E\left[\left|\tau_i(\theta) - \lambda_i C(\alpha(\theta))\right|\right] \\
& = \sum_{i=1}^{n} \left(E\left[\left| -\sum_{j\neq i} \bar{\theta}_j + \bar{c} - c_i \right|,\ i \text{ is pivotal}(+)\right] + E\left[\left| \sum_{j\neq i} \bar{\theta}_j - \bar{c} + c_i \right|,\ i \text{ is pivotal}(-)\right] \right) \\
& \leq \sum_{i=1}^{n} \left(E\left[\left| \sum_{j\neq i} \bar{\theta}_j \right|,\ i \text{ is pivotal}\right] + E\left[c_i,\ i \text{ is pivotal}\right] + E\left[\bar{c},\ i \text{ is pivotal}\right] \right) \\
& \leq \sum_{i=1}^{n} \left(E\left[\left| \sum_{j\neq i} \bar{\theta}_j \right|,\ |\bar{\theta}_i| \geq \left| \sum_{j\neq i} \bar{\theta}_j \right|\right] + E\left[\left| \sum_{j\neq i} \bar{\theta}_j \right|,\ \left| \sum_{j\neq i} \bar{\theta}_j \right| \leq \bar{c}\right] \right. \\
& \qquad + E\left[c_i,\ |\bar{\theta}_i| \geq \left| \sum_{j\neq i} \bar{\theta}_j \right|\right] + E\left[c_i,\ \left| \sum_{j\neq i} \bar{\theta}_j \right| \leq \bar{c}\right] \\
& \qquad \left. + E\left[\bar{c},\ |\bar{\theta}_i| \geq \left| \sum_{j\neq i} \bar{\theta}_j \right|\right] + E\left[\bar{c},\ \left| \sum_{j\neq i} \bar{\theta}_j \right| \leq \bar{c}\right] \right) \\
& \leq \sum_{i=1}^{n} \left(E\left[|\bar{\theta}_i|,\ |\bar{\theta}_i| \geq \left| \sum_{j\neq i} \bar{\theta}_j \right|\right] + \bar{c} P\left(\left| \sum_{j\neq i} \bar{\theta}_j \right| \leq \bar{c} \right) \right. \\
& \qquad + \frac{\alpha \bar{c}}{\underline{w}} E\left[|\bar{\theta}_i|,\ |\bar{\theta}_i| \geq \left| \sum_{j\neq i} \bar{\theta}_j \right|\right] + \frac{\beta \bar{c}}{\underline{w}} P\left(|\bar{\theta}_i| \geq \left| \sum_{j\neq i} \bar{\theta}_j \right| \right) + E[c_i] P\left(\left| \sum_{j\neq i} \bar{\theta}_j \right| \leq \bar{c} \right) \\
& \qquad \left. + \bar{c} P\left(|\bar{\theta}_i| \geq \left| \sum_{j\neq i} \bar{\theta}_j \right| \right) + \bar{c} P\left(\left| \sum_{j\neq i} \bar{\theta}_j \right| \leq \bar{c} \right) \right) \\
& \to 0 \quad \text{as} \quad n \to \infty
\end{aligned}
$$

by Lemma 3, Lemma 5, Lemma 6, Rob (1982, p. 213, Eq 3.4), and Rob (1982, p. 215, Claim 2). ■

Theorem 4. *Let (θ_i, z_i) be a sequence of i.i.d. random vectors for which $z_i \geq \underline{z} > 0$, $E[\theta_i] = \bar{c}$, and $E[\theta_i^2] < \infty$. Suppose*

$$\exists \alpha, \beta \in \mathbb{R}, \forall \theta_i \in \Theta_i,\ E\left[z_i \mid \theta_i\right] \leq \alpha|\theta_i| + \beta.$$

Then given any environment $\mathcal{E} \in \mathbb{E}_B$, the expected distance from ex-post budget balance per person *in the cost-sharing pivot mechanism with cost shares $\lambda_i = \frac{z_i}{\sum_{j=1}^{n} z_j}$ goes to zero as n goes to infinity holding the per person cost $\bar{c} = c/n$ constant, i.e.*

$$\frac{1}{n} E\left[\left|\sum_{i=1}^{n} \tau_i(\theta) - C(\alpha(\theta))\right|\right] \to 0 \quad \textit{as} \quad n \to \infty.$$

Proof. Let $\bar{\theta}_n^* = \max_i \left|\bar{\theta}_i\right|$. We now proceed exactly as in the previous theorem.

$$
\begin{aligned}
&\frac{1}{n} E\left[\left|\sum_{i=1}^{n} \tau_i(\theta) - C(\alpha(\theta))\right|\right] \\
&\leq \frac{1}{n} \sum_{i=1}^{n} E\left[\left|\tau_i(\theta) - \lambda_i C(\alpha(\theta))\right|\right] \\
&\leq \frac{1}{n} \sum_{i=1}^{n} \Bigg(E\left[\left|\bar{\theta}_i\right|,\ \left|\bar{\theta}_i\right| \geq \left|\sum_{j\neq i} \bar{\theta}_j\right|\right] + \bar{c} P\left(\left|\sum_{j\neq i} \bar{\theta}_j\right| \leq \bar{c}\right) \\
&\qquad + \frac{\alpha\bar{c}}{\underline{w}} E\left[|\theta_i|,\ |\theta_i| \geq \left|\sum_{j\neq i} \theta_j\right|\right] + \frac{\beta\bar{c}}{\underline{w}} P\left(|\theta_i| \geq \left|\sum_{j\neq i} \theta_j\right|\right) + E[c_i] P\left(\left|\sum_{j\neq i} \bar{\theta}_j\right| \leq \bar{c}\right) \\
&\qquad + \bar{c} P\left(\left|\bar{\theta}_i\right| \geq \left|\sum_{j\neq i} \bar{\theta}_j\right|\right) + \bar{c} P\left(\left|\sum_{j\neq i} \bar{\theta}_j\right| \leq \bar{c}\right) \Bigg) \\
&\leq E\left[\left|\bar{\theta}_i\right|,\ \left|\bar{\theta}_i\right| \geq \left|\sum_{j\neq i} \bar{\theta}_j\right|\right] + \bar{c} P\left(\left|\sum_{j\neq i} \bar{\theta}_j\right| \leq \bar{c}\right) \\
&\qquad + \frac{\alpha\bar{c}}{\underline{w}} E\left[|\theta_i|,\ |\theta_i| \geq \left|\sum_{j\neq i} \theta_j\right|\right] + \frac{\beta\bar{c}}{\underline{w}} P\left(\theta_n^* \geq \left|\sum_{j\neq i} \theta_j\right|\right) + \bar{c} P\left(\left|\sum_{j\neq i} \bar{\theta}_j\right| \leq \bar{c}\right) \\
&\qquad + \bar{c} P\left(\bar{\theta}_n^* \geq \left|\sum_{j\neq i} \bar{\theta}_j\right|\right) + \bar{c} P\left(\left|\sum_{j\neq i} \bar{\theta}_j\right| \leq \bar{c}\right) \\
&\to 0 \quad \text{as} \quad n \to \infty
\end{aligned}
$$

by Lemma 3, Lemma 4, Rob (1982, p. 212, Lemma 2), and Rob (1982, p. 216, Eq. 3.6). ■

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