XU

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(Major Update Coming Soon)

Abstract

An individual has preferences over experiences. I present axioms which are necessary and sufficient for the existence of an experienced utility (XU) representation in which the utility of an experience is equal to the integral of instantaneous utility over time. I propose a normative principle which states that what is best for society is what an ethical observer would most prefer if they were to *live every life once*. I call this the LELO principle. An ethical observer that respects LELO acts as if they seek to maximize the total XU across individuals. That is, LELO implies utilitarianism with respect to some utility representation and that representation is XU.

1 Introduction

Early economists assumed as primitive a cardinal utility function capturing an individual's intensity of preference.[1] Utilitarian philosophers assume as primitive a similar utility function capturing pleasure, happiness, or satisfaction of desire. But there exist no rigorous foundations for either. It is not clear how to construct these functions nor precisely what they capture. I propose an axiomatic theory of experienced utility, which simultaneously produces a theory of preference intensity over alternatives and a theory of instantaneous utility experienced over time, uniting and grounding these two concepts.

The main idea is simple. I consider alternatives as being experienced over time, and I consider preferences over these experiences. I characterize preferences for which there exists an *experienced utility* (XU) representation in which the utility of an experience is equal to the integral of instantaneous utility over time. By leveraging

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[1] Later, intensity of preference was eschewed in favor of purely ordinal preference.

an individual's ability to compartmentalize an otherwise indivisible alternative into separate experiences, I am able to extract her intensity of preference at each moment in time and for the alternative overall.

I propose a normative principle which states that what is best for society is what an ethical observer would most prefer if they were to *live every life once*. I call this the LELO principle. I consider a social planner who respects LELO. Such a planner's preference over alternatives is represented by the sum of the experienced utilities across individuals. That is, LELO implies utilitarianism with respect to some utility representation and that representation is XU.

To understand the basic approach, consider the following simple example. Let $X = \{a, b\}$ be a set of alternatives, where a represents an "apple" and b a "banana". Standard theory assumes an individual has preferences $\succeq$ over X, where $a \succeq b$ is taken to mean that the individual prefers to *consume* an apple to a banana. Hence, a actually represents an experience—the experience of consuming an apple—that occurs over time. With this in mind, it is natural to consider experiences over subintervals of time—for instance, the first bite of an apple. Let $\mathcal{I}$ be the set of intervals on $[0, 1]$. An *experience* is a pair (x, I), where x is an alternative and I is an interval. Suppose it takes two minutes to eat an apple. Then $(a, [0, 1))$ is the complete experience of eating an apple, $(a, [0, .5))$ is the experience of just the first minute, and $(a, [.5, 1))$ is the experience of just the second.[2] Within this framework, standard theory assumes an individual has preferences over *complete* experiences. This reveals the order of her preference on X, but not the intensity. Suppose instead that an individual has preferences $\succeq$ over (all) experiences $X \times \mathcal{I}$. Such preferences provide enough information to construct an intensity of preference on X. In particular, I present four axioms that are necessary and sufficient for the existence of an XU representation $U(x, I) = \int_I v_x(t)\ \mathrm{d}t$, where $v_x : [0, 1] \to \mathbb{R}$ is an instantaneous utility function for each alternative capturing the individual's intensity of preference at each moment in time. The representation is unique up to a positive scalar, i.e. U is a ratio scale.[3]

[2] See Section 3 for a more precise definition of an experience.

[3] $\tilde{U}$ is another XU representation if and only if there exists an $\alpha > 0$ such that $\tilde{U} = \alpha U$.

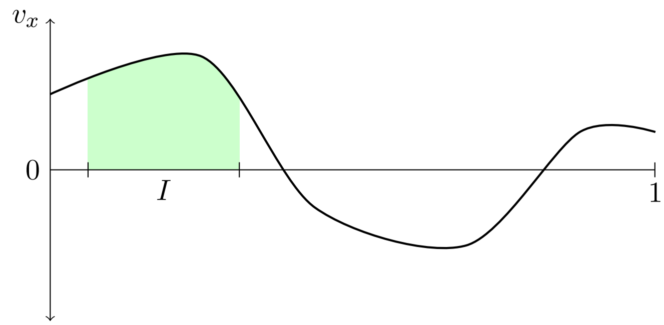

Figure 1: The *experienced utility* of (x, I) is given by the area in green.

In a ratio scale representation, utility signs and ratios are meaningful.[4] That is, they convey something explicit about the primitive. Suppose that consuming an apple gives positive experienced utility, i.e. $U(a, [0, 1)) > 0$. This *means* that $(a, [0, 1)) \succ (a, [0, 0))$. The experience of consuming an apple is better than experiencing nothing. Suppose in addition that consuming a banana gives half the experienced utility of consuming an apple, i.e. $.5U(a, [0, 1)) = U(b, [0, 1))$. Given U's integral form, there exists a $k \in (0, 1)$ such that $U(a, [0, k)) = U(a, [k, 1)) = .5U(a, [0, 1))$. That is, the experience of consuming an apple can be cut in “half”, not by time (or volume), but by *preference*. For instance, maybe the first 30 seconds of consuming an apple is equally as good as the final 90. Then $k = .25$. A banana is “half as good” as an apple *means* that “half” the experience of an apple, by *preference* (the first 30 or final 90 seconds), is indifferent to the complete experience of a banana, i.e. $(a, [0, .25)) \sim (a, [.25, 1)) \sim (b, [0, 1))$.

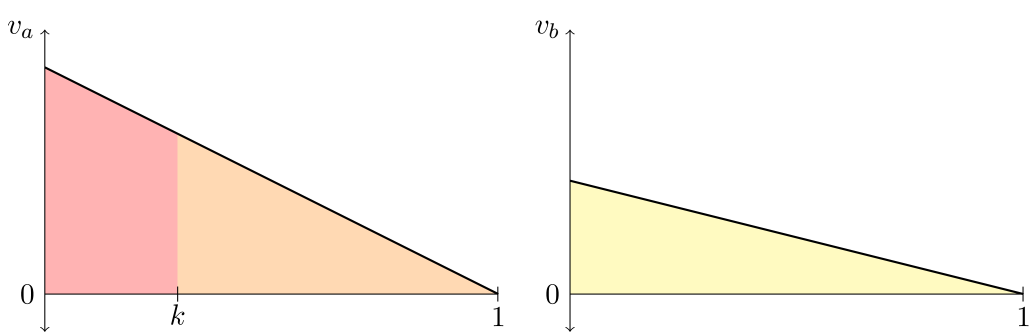

Figure 2: The area in red, orange, and yellow are equal. The banana (right) gives half the utility of the apple (left). Half the experience of the apple, by preference, is indifferent to the complete experience of the banana.

[4]Comparisons of utility levels (ordinal comparisons) and differences (cardinal comparisons) are meaningful as well.

Notice that the primitive (preferences over experiences) contains no notion of sign or intensity of preference. Nevertheless, the representation reveals an intuitive and consistent way to deduce such characteristics from the primitive. That is, the representation provides an intuitive way to define what it means to have (signed) intensities of preference, and the axioms provide sufficient conditions for such properties to exist and be internally consistent.[5]

The key axiom giving rise to this representation is *temporal monotonicity*. The term was introduced in Kahneman et al. (1993) as a normative principle for evaluating experiences and states that "adding moments of pain to the end of an episode can only make the episode worse, and that adding moments of pleasure must make it better." The axiom is a formalization of this notion. It states that for any interval $I = I_1 \cup I_2$, where I_1 and I_2 are disjoint, and $J = J_1 \cup J_2$, where J_1 and J_2 are disjoint, if (x, I_1) is preferred to (y, J_1) and (x, I_2) is preferred to (y, J_2), then (x, I) is preferred to (y, J). For example, if the first 30 seconds of consuming an apple is preferred to the first minute of consuming a banana,[6] and the last 90 seconds of consuming an apple is preferred to the last minute of consuming a banana, then it must be that the complete experience of consuming an apple is preferred to the complete experience of consuming a banana.

The rest of the paper is organized as follows. Section 2 provides some background and related literature. Section 3 considers preferences over experiences with a single alternative. Section 4 considers preferences over experiences with an arbitrary set of alternatives. Section 5 considers preferences over collections of experiences with arbitrary alternatives. Section 6 uses the theory developed in Section 5 as a foundation for utilitarianism. Section 7 concludes. Appendix A defines a formal theory of measurement. Appendix B elaborates on the interpretation of preferences over experiences. Appendix C contains all proofs.

2 Background and Related Literature

A cardinal utility function capturing intensity of preference was once considered a primitive in economic theory. Later, economists realized that ordinal preferences were sufficient for much of their analyses and abandoned cardinal utility as a primitive in favor of preferences. A utility function was then simply taken to be a numerical representation of preferences, and so only contained *ordinal* information. With the onset of expected utility, cardinal utility found its way back into economic theory. This time, though, utility itself was not assumed to be primitive. Like ordinal utility, expected utility is merely a representation of preferences.

[5]Given the axioms, the sign of an experience and the intensity of preference between a pair of experiences is uniquely identified by $\succeq$.

[6]Suppose it takes two minutes to eat a banana.

How can a representation of ordinal preferences contain *cardinal* meaning? First, preferences are expanded onto a richer domain (from the set of alternatives to the set of probability distributions over alternatives). Second, a particular mapping from utilities on the original domain (Bernoulli utilities) to utilities on the expanded domain (von-Neumann-Morgenstern utilities) is posited—the von-Neumann-Morgenstern utility of a probability distribution over alternatives should be the probability weighted sum of the Bernoulli utilities of the alternatives. Given such a representation, comparisons of *Bernoulli utility differences* are meaningful. If $u(a) - u(b) > u(c) - u(d)$, then $.5u(a) + .5u(d) > .5u(b) + .5u(c)$ and so a 50/50 lottery between a and d is strictly preferred to a 50/50 lottery between b and c.

Notice that the meaning captured by a Bernoulli utility function differs from the meaning primitive cardinal utility functions were assumed to capture. In particular, a Bernoulli utility function captures an individual's attitude to *risk*, while a primitive cardinal utility function captures an individual's *intensity* of preference. Equating the two is a longstanding misconception in economic theory, plainly identified by Luce and Raiffa (1957) as "Fallacy 3" in their list of common misinterpretations of expected utility. A similar story can be told when expanding the domain of preferences from a set of alternatives to the set of *streams* of alternatives[7] (as in intertemporal choice, see Koopmans (1960); Bleichrodt, Rohde and Wakker (2008)) or to the set of alternatives and *money* (as in willingness to pay). Both result in a cardinal representation on the original preference domain that captures meaning about the richer domain (and hence does not capture preference intensity per se).

A final approach expands the domain of preferences from a set of alternatives to the set of *pairs* of alternatives (see Suppes and Winet (1955); Köbberling (2006)). Preferences over this richer domain are taken to represent preferences over *preference differences*. That is, $ab \succeq cd$ means that the *difference* in preference between a and b is larger than the *difference* in preference between c and d. This results in a cardinal representation on alternatives that, indeed, can be taken to capture preference intensity. But this is not so different from assuming cardinal utility itself as primitive, and the issues with doing so here are the same. In many ways, this is the crux of what is objectionable about taking cardinal utility as primitive—that comparisons of preference differences are considered meaningful at the outset.

This paper proposes the first theory of cardinal utility capturing preference intensity that does not take preference intensity itself as primitive. By expanding the domain of preferences from a set of alternatives to the set of alternatives *and* intervals (inter-

[7]Notice the difference between a *stream* of alternatives and alternatives which are experienced *over time*. For both, let time be continuous but finite in length. Let X be an arbitrary set of alternatives. A *stream* of alternatives maps points in time to alternatives. In intertemporal choice, an individual has preferences over all such functions $X^{[0,1]}$. An *experience* is an alternative and an interval of time. Here, an individual has preferences over all such pairs $X \times \mathcal{I}$.

preted as experiences), I am able to construct an experienced utility representation that captures what it *means* for an individual to have intensity of preference.

The idea of experienced utility is not new. Kahneman, Wakker and Sarin (1997) were first to introduce the term, but the idea goes back to Edgeworth's hedonometer. In his 1881 book *Mathematical Psychics*, Edgeworth (p. 101) proposed the idea of measuring pleasure continuously over time: "let there be granted to the science of pleasure what is granted to the science of energy; to imagine an ideally perfect instrument, a psychophysical machine, continually registering the height of pleasure experienced by an individual. . . . the quantity of happiness between two epochs is represented by the area contained between the zero-line, perpendiculars thereto at the points corresponding to the epochs, and the curve traced by the index." This hypothetical instrument came to be known as a "hedonometer" and the notion of instantaneous pleasure as "hedonic flow". Despite being introduced over a century ago, no formal theory of measurement[8] of hedonic flow has yet been proposed. Without a formal theory, such a measure has no meaning. In other words, hedonic flow is a *conjecture*: the conjecture that there exists a measure assigning numbers to instants of time such that the precise meaning we prescribe to the measure allows it to be intuitively understood as "height of pleasure". This paper proposes such a theory.

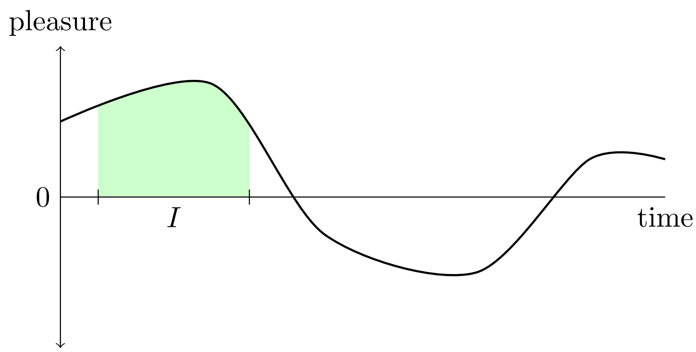

Figure 3: The *quantity of happiness* of I is given by the area in green.

3 A Theory of Measurement of Hedonic Flow

In this section I consider preferences over experiences with a single alternative, yielding a theory of measurement of instantaneous preference intensity or hedonic flow. The primitive is a tuple $(\mathcal{I}, \succeq)$, where $\mathcal{I}$ is the set of all left-closed, right-open intervals[9] on $[0, 1]$ and $\succeq$ is a preference relation on $\mathcal{I}$. For any $I, J \in \mathcal{I}$ such that

[8]See Appendix A for a discussion of measurement theory.

[9]I consider left-closed, right-open intervals so that adjacent intervals are disjoint and the union of adjacent intervals is itself an interval.

I and J are adjacent (and hence also disjoint), define $I \oplus J \equiv I \cup J$. Notice that $\mathcal{I}$ is closed under $\oplus$.

An *alternative* is a complete description of the state of the world over a finite period of time, represented by $[0,1)$. An *experience* is the complete description of the state of the world over any subinterval of $[0,1)$. Crucially, these descriptions include the *state of mind* of the individual who experiences it. For example, if $[0,1)$ is the experience of consuming an apple, then it must specify how hungry the individual feels at each point in time. The relation $\succeq$ captures the individual's preferences for *experiencing* these states as they are described. For instance, if $[0,.5)$ is the first minute of eating an apple, during which time the individual feels hungry, and $[.5,1)$ is the experience of the second minute of eating an apple, during which time the individual feels full, then $[0,.5) \succ [.5,1)$ means that the individual would prefer to experience the state of the world $[0,.5)$, so described, to $[.5,1)$.[10]

The first axiom simply identifies the primitive.

Axiom 1.1 (Rationality)**.** An individual has preferences $\succeq$ over experiences $\mathcal{I}$.

It will prove useful to represent intervals as points in $\mathbb{R}^2$, where the interval $[x,y)$ is represented by the point (x,y). $\mathcal{I}$ is then represented by the set of points above the diagonal in the unit square. As in consumer theory, indifference curves will play an important role in the analysis. However, they will not look anything like the classical indifference curves from consumer theory.

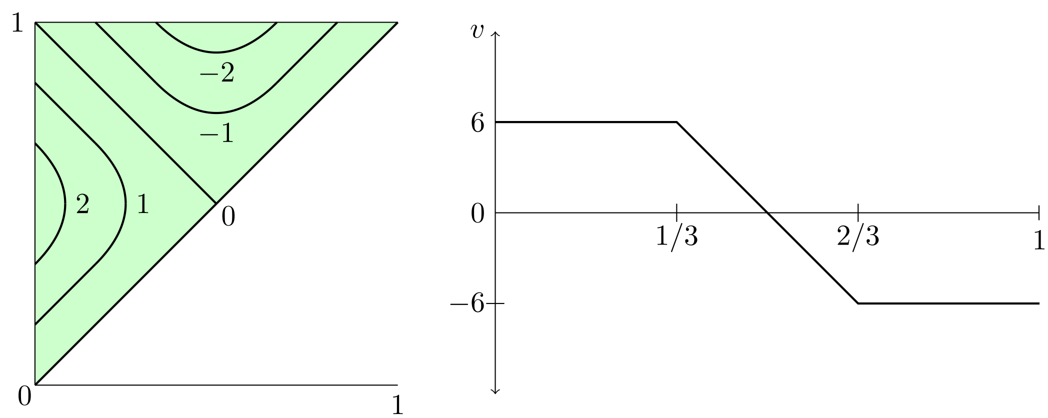

Figure 4: $\mathcal{I}$ is represented by the shaded triangle. Note that each point on the diagonal represents the (same) empty interval. Consider a preference on $\mathcal{I}$ represented by $U(I) = \int_I v(t)\ \mathrm{d}t$, with v shown on the right. Some indifference curves of these preferences are plotted on the left.

[10]See Appendix B for an elaboration on the interpretation of preferences over experiences.

The second axiom provides the conceptual backbone for the representation theorem.

Axiom 1.2 (Temporal Monotonicity)**.** For any $I, J, K, L \in \mathcal{I}$, if $I \oplus K$ and $J \oplus L$ are defined, then $I \succeq J$ and $K \succeq L$ implies $I \oplus K \succeq J \oplus L$. Moreover, $I \succ J$ and $K \succeq L$ implies $I \oplus K \succ J \oplus L$.

In other words, if I is preferred to J, and we concatenate to I an experience which is preferred to the experience we concatenate to J, then the preference is unchanged. A common concern is that by concatenating I to K, we change the experience of the "K portion" of $I \oplus K$. But this is not the case. Since the individual's state of mind is part of the description of the experience itself, it remains the same whether she is experiencing I only, K only, or $I \oplus K$.

The third axiom will be familiar from consumer theory. In fact, it is precisely the same condition.

Axiom 1.3 (Continuity)**.** For any $I, J \in \mathcal{I}$, if $I \succ J$, then there exists an $\varepsilon > 0$ such that if $I' \in \mathcal{I}$ is less than ε distant from I and $J' \in \mathcal{I}$ is less than ε distant from J, then $I' \succ J'$.

To make this precise, I must clarify how to measure distance between intervals. In consumer theory, we often use Euclidean distance. This (and many other notions) would suffice, however the following measure of distance has a nice interpretation in our setting. Define the distance between $I = [I_0, I_1)$ and $J = [J_0, J_1)$ by $\max\{|I_0 - J_0|, |I_1 - J_1|\}$. Under this definition, continuity states that small changes in the *interval*, defined as small changes to the start and end time, result in small changes in *preference*. In $\mathbb{R}^2$, the set of points less than ε distant from I forms a square of width 2ε around I (instead of a circle of radius ε with Euclidean distance).

The final axiom is technical and will take some effort to develop. The following objects are depicted graphically in Figure 5. Given a start time a, let $Y_{xy}(a)$ be the set of all end times b such that $(a, b) \sim (x, y)$. Given an end time b, let $X_{xy}(b)$ be the set of all start times a such that $(a, b) \sim (x, y)$. Let Φ_{xy} be the set of indifference curves through the point (x, y) that are *closest* to the horizontal line y. Let Γ_{xy} be the set of inverse indifference curves through the point (x, y) that are *closest* to the vertical line x.[11]

[11]Formally, let $N(t, w) \equiv [t - w, t + w] \cap [0, 1]$ denote a neighborhood of width w around t within $[0, 1]$. For any $x < y$, let $Y_{xy}(a) \equiv \{b \in [0, 1] : (x, y) \sim (a, b)\}$ and $\Phi_{xy} \equiv \{\phi_{xy} \in [0, 1]^{[0,1]} : \exists w > 0,\ \forall a \in N(x, w),\ \phi_{xy}(a) \in Y_{xy}(a)$ and $\nexists\, b \in Y_{xy}(a),\ |y - b| < |y - \phi_{xy}(a)|\}$. Similarly, let $X_{xy}(b) \equiv \{a \in [0, 1] : (x, y) \sim (a, b)\}$ and $\Gamma_{xy} \equiv \{\gamma_{xy} \in [0, 1]^{[0,1]} : \exists w > 0,\ \forall b \in N(y, w),\ \gamma_{xy}(b) \in X_{xy}(b)$ and $\nexists\, a \in X_{xy}(b),\ |x - a| < |x - \gamma_{xy}(b)|\}$.

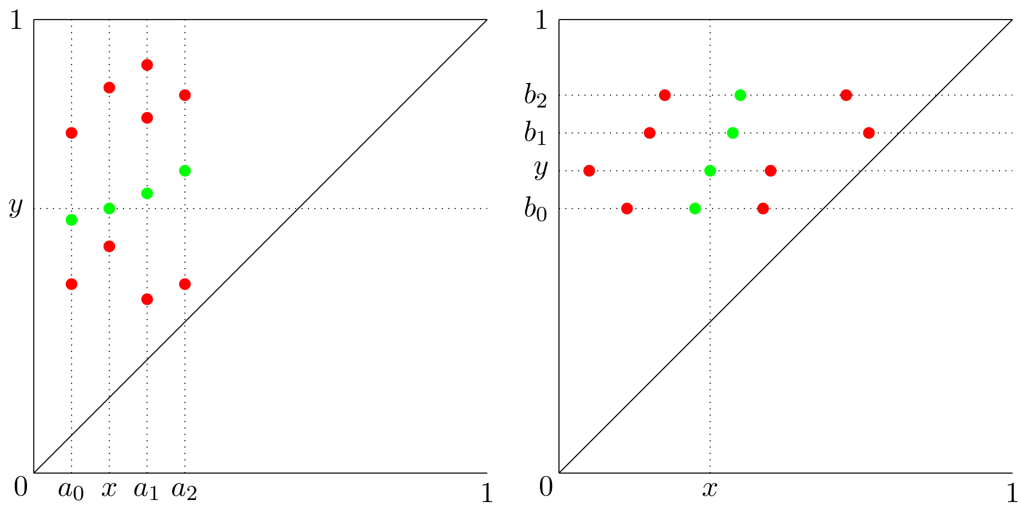

Figure 5: *Left:* For each $a \in \{a_0, x, a_1, a_2\}$, the set of points vertically above denotes $Y_{xy}(a)$ and the green point denotes $\hat{\phi}_{xy}(a)$ for some $\hat{\phi}_{xy} \in \Phi_{xy}$. *Right:* For each $b \in \{b_0, b_1, y, b_2\}$, the set of points horizontally across denotes $X_{xy}(b)$ and the green point denotes $\hat{\gamma}_{xy}(b)$ for some $\hat{\gamma}_{xy} \in \Gamma_{xy}$.

Definition 1.1. Preferences $\succeq$ are *non-trivial* if there exist $I, J \in \mathcal{I}$ such that $I \succ J$.

Axiom 1.4 (Smoothness)**.** If preferences are non-trivial,[12] there exists $\tau \in (0,1)$ such that for all $x \in [0, \tau)$ and $y \in (\tau, 1]$, there exist continuous indifference curves $\phi_{x\tau}(\cdot) \in \Phi_{x\tau}$ and $\gamma_{\tau y}(\cdot) \in \Gamma_{\tau y}$ such that the function $\hat{v} : [0,1] \to \mathbb{R}$ defined by

$$\hat{v}(t) = \begin{cases} \phi'_{t\tau}(t) & \text{if } 0 \leq t < \tau \\ 1 & \text{if } t = \tau \\ \gamma'_{\tau t}(t) & \text{if } \tau < t \leq 1 \end{cases}$$

is continuous and crosses zero finitely many times.[13]

This is a bit of a mouthful. There exists a line from $(0, \tau)$ to (τ, τ) to $(\tau, 1)$ for some $\tau \in (0,1)$, such that for each point (x, τ) on the horizontal portion of the line, there exists an indifference curve $\phi_{x\tau}(\cdot) \in \Phi_{x\tau}$ closest to τ that is continuous in a neighborhood of and differentiable at (x, τ), and for each point (τ, y) on the vertical portion of the line, there exists an inverse indifference curve $\gamma_{\tau y}(\cdot) \in \Gamma_{\tau y}$ closest to τ that is continuous in a neighborhood of and differentiable at (τ, y). Moreover, as we

[12] The subsequent condition implies that preferences are non-trivial.

[13] Let $Z = \{z \in [0,1] : \hat{v}(z) = 0 \text{ and } \nexists \delta > 0, \forall z' \in (z - \delta, z + \delta),\ \hat{v}(z') = 0\}$. Then $\hat{v}$ crosses zero finitely many times if Z is finite.

trace along the line, the slopes of these indifference curves must form a continuous function $\hat{v} : [0,1] \to \mathbb{R}$ that crosses zero finitely many times, and $\hat{v}(\tau) = 1$.

What does the axiom say about the underlying primitive? Consider the point (x, y). Suppose I perturb the start time by ε. To keep the individual indifferent, I must then perturb the end time to $\phi_{xy}(x + \varepsilon)$. Requiring continuity of ϕ_{xy} in a neighborhood of x means that small changes to the start time around x require small changes to the end time to keep the individual indifferent. Requiring differentiability of ϕ_{xy} at x means that small changes to x require *directly proportional* changes to y to keep the individual indifferent. Similarly for γ_{xy}, except with the start and end times reversed. Continuity of $\hat{v}$ means that small shifts in the interval of consideration along the line connecting $(0, \tau)$, (τ, τ), and $(\tau, 1)$ result in small changes in these slopes, and since $\hat{v}(\tau) = 1$, the slopes around the empty interval $[\tau, \tau)$ should be close to one, capturing the idea that moments in time arbitrarily close together are arbitrarily close in preference. Moreover, $\hat{v}$ cannot be infinitely 'squiggly', as it must cross zero only finitely many times. This is depicted in Figure 6.

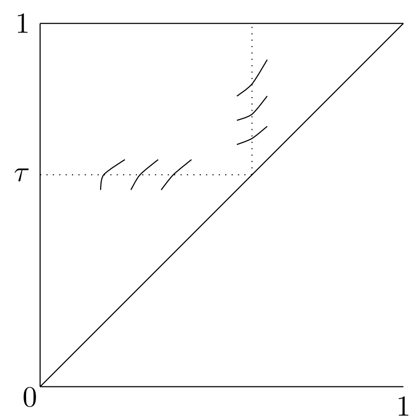

Figure 6: The line connecting $(0, \tau)$, (τ, τ), and $(\tau, 1)$ and the indifference curves through several points on the line. Axiom 1.4 requires that the slopes of these indifference curves form a continuous function $\hat{v}$ (that crosses zero finitely many times and satisfies $\hat{v}(\tau) = 1$).

It is apparent that the axiom does not impose any substantial or controversial conditions on the individual's preferences over experiences and that it is simply a technical condition arising from the smooth structure of the representation we desire. To be clear, we seek a representation of the form $U(I) = \int_I v(t)\ \mathrm{d}t$, and for simplicity and elegance I, the researcher, *choose* to require that v be sufficiently "nice". In particular, I require that it be continuous and that it cross zero finitely many times.[14]

[14] I require the latter condition because it is helpful for the proof as I have constructed it, and I feel it is substantively innocuous. It may be possible to prove the corresponding theorem without

Given this, Axiom 1.4 is immediately necessary. If a representation of the form $U(I) = \int_I v(t)\ \mathrm{d}t$ exists and $v(\tau) \neq 0$,[15] then the slope of the indifference curve at (x, τ) is $v(x)/v(\tau)$ and the inverse slope of the indifference curve at (τ, y) is $v(y)/v(\tau)$. Hence $\hat{v} = v/v(\tau)$, and the axiom follows.

We are now ready for the representation theorem.

Definition 1.2. A preference relation $\succsim$ on a set X is *represented* by a utility function $u : X \to \mathbb{R}$ if for any $x, y \in X$, $x \succsim y \iff u(x) \geq u(y)$.

Theorem 1. *Axioms 1.1, 1.2, 1.3, and 1.4 hold if and only if there exists an instantaneous utility function $v : [0,1] \to \mathbb{R}$ that is continuous and crosses zero finitely many times, such that $\succeq$ is represented by the experienced utility function $U : \mathcal{I} \to \mathbb{R}$ given by*

$$U(I) = \int_I v(t)\ \mathrm{d}t.$$

Moreover, $\tilde{U}(I) = \int_I \tilde{v}(t)\ \mathrm{d}t$ is another representation if and only if $\tilde{v} = \alpha v$ for some $\alpha > 0$.

4 A Theory of Measurement of Preference Intensity

In this section I consider preferences over experiences with an arbitrary set of alternatives, yielding a theory of measurement of preference intensity. The primitive is a tuple $(X \times \mathcal{I}, \succeq)$, where X is an arbitrary set of alternatives, $\mathcal{I}$ is the set of all left-closed, right-open intervals on $[0,1]$, and $\succeq$ is a preference relation on $X \times \mathcal{I}$. For any $x \in X$ and $I, J \in \mathcal{I}$ such that I and J are adjacent, define $(x, I) \oplus (x, J) \equiv (x, I \cup J)$. Notice that $X \times \mathcal{I}$ is closed under $\oplus$.

As before, an *alternative* is a complete description of the state of the world over a finite period of time, and, given an alternative, an *experience* is the complete description of the state of the world over any subinterval of time. In the previous section we considered a single alternative, so an experience was simply an interval $I \in \mathcal{I}$. Now, we consider an arbitrary set of alternatives, so an experience is a pair $(x, I) \in X \times \mathcal{I}$ specifying both an alternative and an interval.

We seek the same representation as in Theorem 1, except with an instantaneous utility function v_x *for each* alternative $x \in X$. It is worth pointing out that this result does not follow immediately from Theorem 1—in particular, because an experience cannot span multiple alternatives. For example, the individual cannot compare the second minute of eating an apple *and* the first minute of eating a banana to the

this condition.

[15]If such a representation exists and preferences are non-trivial, then there exists $\tau \in (0,1)$ such that $v(\tau) \neq 0$.

complete experience of eating an apple.[16] Figure 7 contains a simple illustration of this point.

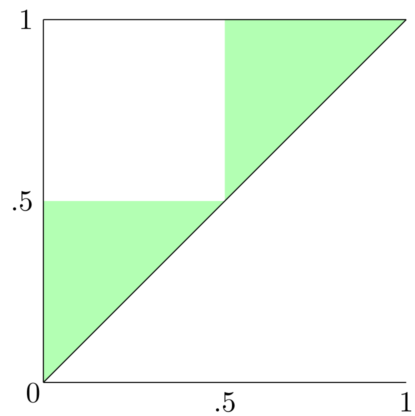

Figure 7: Let $X = \{x, y\}$ and suppose we "stack" the timelines $[0,1]_x$ and $[0,1]_y$ in hopes of applying Theorem 1. That is, on the new timeline $(x, [a, b))$ is represented by $[a/2, b/2)$ and $(y, [c, d))$ is represented by $[.5 + c/2, .5 + d/2)$. With preferences over such intervals, we could apply Theorem 1 and be done; but we do not have preferences over *all* such intervals. We have preferences only over points in the green triangles.

While not immediate, the desired result holds (with a small caveat). The following axioms are simple extensions of the original axioms to this environment, along with a (trivial) consistency axiom.

Axiom 2.1 (Rationality)**.** An individual has preferences $\succeq$ over experiences $X \times \mathcal{I}$.

Axiom 2.2 (Temporal Monotonicity)**.** For any $x, y \in X$ and $I, J, K, L \in \mathcal{I}$, if $(x, I \oplus K)$ and $(y, J \oplus L)$ are defined, then $(x, I) \succeq (y, J)$ and $(x, K) \succeq (y, L)$ implies $(x, I \oplus K) \succeq (y, J \oplus L)$. Moreover, $(x, I) \succ (y, J)$ and $(x, K) \succeq (y, L)$ implies $(x, I \oplus K) \succ (y, J \oplus L)$.

Axiom 2.3 (Continuity)**.** For any $(x, I), (y, J) \in \mathcal{I}$, if $(x, I) \succ (y, J)$, then there exists an $\varepsilon > 0$ such that if $I' \in \mathcal{I}$ is less than ε distant from I and $J' \in \mathcal{I}$ is less than ε distant from J, then $(x, I') \succ (y, J')$.

Axiom 2.4 (Smoothness)**.** For each $x \in X$, $\succeq$ over experiences containing x satisfies Axiom 1.4.

[16] We will consider such comparisons in Section 5.

Axiom 2.5 (Consistent Null Sets)**.** Empty experiences are indifferent across alternatives. That is, for any $x, y \in X$, $(x, \emptyset) \sim (y, \emptyset)$.

The last axiom is, in a sense, just semantics. We would like to consider $(x, \emptyset)$ and $(y, \emptyset)$ to be identical experiences, since both represent the *lack* of an experience. However, since they are not formally equivalent, we must assume indifference explicitly.[17]

These axioms give rise to the desired representation. However, for a certain class of preferences, the representation is *not* a ratio scale.

Definition 2.1. A preference $\succeq$ over $X \times \mathcal{I}$ is *diametric* if for each alternative $x \in X$, either x is *uniformly positive*, i.e. $(x, I) \succeq (x, \emptyset)$ for all $I \in \mathcal{I}$, or it is *uniformly negative*, i.e. $(x, I) \preceq (x, \emptyset)$ for all $I \in \mathcal{I}$, and there exists at least one uniformly positive and uniformly negative alternative.

Let $X^+ = \{x \in X : \exists I \in \mathcal{I}, (x, I) \succ (x, \emptyset)\}$ denote the set of alternatives with some positive experience and $X^- = \{x \in X : \exists I \in \mathcal{I}, (x, I) \prec (x, \emptyset)\}$ denote the set of alternatives with some negative experience. Note that $\succeq$ is diametric if and only if X^+ and X^- are non-empty and disjoint.

Theorem 2. *Axioms 2.1, 2.2, 2.3, 2.4, and 2.5 hold if and only if there exists a set $V = \{v_x\}_{x \in X}$ of instantaneous utility functions $v_x : [0,1] \to \mathbb{R}$, each of which is continuous and crosses zero finitely many times, such that $\succeq$ is represented by the experienced utility function $U : X \times \mathcal{I} \to \mathbb{R}$ given by*

$$U(x, I) = \int_I v_x(t) \; \mathrm{d}t.$$

If preferences are diametric, $V^+ = \{v_x\}_{x \in X^+}$ and $V^- = \{v_x\}_{x \in X^-}$ are independently ratio-scale.[18] *Otherwise, V is ratio-scale.*

The inability to pin down a ratio scale under diametric preferences stems from the lack of non-trivial comparisons between alternatives of different sign. Suppose x is uniformly positive and y is uniformly negative. Then $(x, I) \succeq (y, J)$ for all $I, J \in \mathcal{I}$ and strictly if I or J is non-empty. Hence, we may scale v_x and v_y independently without changing the underlying preference. This is not the case if for some $\hat{I} \in \mathcal{I}$, $(x, \hat{I}) \prec 0$ (so that preferences are not diametric), since changing the scale of v_x independently of v_y changes the ranking between $(x, \hat{I})$ and (y, J), for J non-empty. Furthermore, this is not the case if the individual can rank *collections* of experiences across alternatives, since changing the scale of v_x independently of v_y changes the

[17] In the setting of Theorem 1, the analogous axiom is $[t, t) \sim [t', t')$ for any $t, t' \in [0, 1]$. But this is not necessary since $[t, t) = [t', t') = \emptyset$, i.e. the objects we would like to be identical are identical.

[18] $\tilde{U}(x, I) = \int_I \tilde{v}_x(t) \; \mathrm{d}t$ is another representation if and only if for some $\alpha_1, \alpha_2 > 0$, $\tilde{v}_x = \alpha_1 v_x$ for all $x \in X^+$ and $\tilde{v}_x = \alpha_2 v_x$ for all $x \in X^-$.

ranking between the collection "$(x, I)+(y, J)$" and the empty experience. I consider such preferences next.

5 A(nother) Theory of Measurement of Preference Intensity

In this section I consider preferences over collections of experiences with an arbitrary set of alternatives, yielding a theory of measurement of preference intensity. Formally, I consider preferences over *multisets*[19] of experiences (allowing for repetitions). Denote the set of all finite multisets containing only elements of $X \times \mathcal{I}$ by $\mathcal{M}(X \times \mathcal{I})$. When the meaning is clear from context, I will sometimes write (x, I) to mean the singleton $\{(x, I)\}$. The primitive is a tuple $(\mathcal{M}(X \times \mathcal{I}), \succeq)$, where $\succeq$ is a preference relation on $\mathcal{M}(X \times \mathcal{I})$. Multisets have a natural concatenation operation commonly denoted by $+$. For any $A, B \in \mathcal{M}(X \times \mathcal{I})$, define $A + B$ as the multiset with multiplicities summed across multisets. For example, $\{(a, I), (b, J)\} + \{(b, J), (c, K)\} = \{(a, I), (b, J), (b, J), (c, K)\}$. Similarly, define $A - B$ as the multiset with multiplicities subtracted across multisets, so that $\{(a, I), (b, J), (b, J), (c, K)\} - \{(b, J), (c, K)\} = \{(a, I), (b, J)\}$. Notice that $\mathcal{M}(X \times \mathcal{I})$ is closed under $+$.

We seek a representation in which the experienced utility of a multiset is given by the sum of the experienced utilities of its elements. The following axioms are simple extensions of those in Theorem 2 to this environment, along with an additional (trivial) consistency axiom.

Axiom 3.1 (Rationality)**.** An individual has preferences $\succeq$ over $\mathcal{M}(X \times \mathcal{I})$.

Axiom 3.2 (Monotonicity)**.** For any $A, B, C, D \in \mathcal{M}(X \times \mathcal{I})$, if $A \succeq B$ and $C \succeq D$ then $A + C \succeq B + D$. Moreover, if $A \succ B$ and $C \succeq D$ then $A + C \succ B + D$.

In the previous two sections, we considered preferences over *experiences* and used interval concatenation $\oplus$ to concatenate two adjacent experiences into a third experience. Here, we consider preferences over *collections* of experiences and use multiset concatenation $+$ to concatenate any two collections into a third collection. Axiom 3.2 reflects this change. While Axioms 1.2 and 2.2 have a clear temporal component, Axiom 3.2 does not. Hence, I refer to it simply as *monotonicity.* It states that if a collection of experiences A is preferred to B, and a collection of experiences C

[19] A multiset is a collection of objects in which order is ignored (like a set) but multiplicity is not (unlike a set). I denote a multiset by $\{\cdot\}$. Then $\{a, b\} = \{b, a\} \neq \{a, a, b, b\}$. The cardinality of a multiset is equal to the sum of the multiplicities of its elements, e.g. $|\{a, a, b, b\}| = 4$. Notice that with standard sets, $\{(x, [0, .5))\}$ is equal to $\{(x, [0, .5)), (x, [0, .5))\}$ is *not* equal to $\{(x, [0, .5)), (x, [0, .50\ldots01))\}$. In this sense, sets allow us to get arbitrarily close to a repeated experience, without allowing the repeated experience itself.

is preferred to D, then the combination of the two preferred collections $A + C$ is preferred to the combination of the other two, $B + D$. When paired with Axiom 3.6, Axiom 3.2 implies Axiom 2.2.[20]

Axiom 3.3 (Continuity)**.** For any $A, B \in \mathcal{M}(X \times \mathcal{I})$, if $A \succ B$, then for any $(x, I) \in A$ and $(y, J) \in B$ there exists an $\varepsilon > 0$ such that if $I' \in \mathcal{I}$ is less than ε distant from I and $J' \in \mathcal{I}$ is less than ε distant from J, then $A - (x, I) + (x, I') \succ B - (y, J) + (y, J')$.

Axiom 3.4 (Smoothness)**.** For each $x \in X$, $\succeq$ over singletons containing x satisfies Axiom 1.4.

Axiom 3.5 (Consistent Null Sets)**.** Empty experience singletons are indifferent across alternatives. That is, for any $x, y \in X$, $\{(x, \emptyset)\} \sim \{(y, \emptyset)\}$.

Axiom 3.6 (Consistent Concatenations)**.** For any $x \in X$, if $I, J \in \mathcal{I}$ are adjacent, then $\{(x, I)\} + \{(x, J)\} \sim \{(x, I \oplus J)\}$.

As before, the final axiom is essentially just semantics. We would like to consider $\{(x, I)\} + \{(x, J)\}$ and $\{(x, I \oplus J)\}$ to be the same object, but since they are not formally equivalent, we must assume indifference explicitly. We can view this as imposing consistency between multiset-concatention $+$ and interval-concatenation $\oplus$.

These axioms yield the desired result (even if preferences are diametric).

Theorem 3. *Axioms 3.1, 3.2, 3.3, 3.4, 3.5, and 3.6 hold if and only if there exists a set $V = \{v_x\}_{x \in X}$ of instantaneous utility functions $v_x : [0, 1] \to \mathbb{R}$, each of which is continuous and crosses zero finitely many times, such that $\succeq$ is represented by the experienced utility function $\mathcal{U} : \mathcal{M}(X \times \mathcal{I}) \to \mathbb{R}$ given by*

$$\mathcal{U}(A) = \sum_{(x,I) \in A} \int_I v_x(t) \, \mathrm{d}t = \sum_{(x,I) \in A} U(x, I).$$

Moreover, V is ratio-scale.

6 A Foundation for Utilitarianism

In this section, I use the framework of the previous section to build a foundation for utilitarianism. Broadly speaking, utilitarianism states that a social state is ethically

[20] Let I, J and K, L be adjacent intervals in $\mathcal{I}$. For any $x, y \in X$, suppose $(x, I) \succeq (y, J)$ and $(x, K) \succeq (y, L)$. Then $\{(x, I)\} + \{(x, K)\} \succeq \{(y, J)\} + \{(y, L)\}$ by Axiom 3.2 and $(x, I \oplus K) \succeq (x, J \oplus L)$ by Axiom 3.6.

preferred to another if and only if it produces greater total "utility", where total utility is the sum of individual utilities, and individual utility represents pleasure, happiness, or satisfaction of desire. But where exactly do these utilities come from? What precisely do they capture? And what does it mean to add them across individuals? A formal theory of utilitarianism must provide a theory of measurement of individual utilities (i.e., a theory of measurement of individual pleasure, etc.[21]) and justify why a social planner should seek to maximize their sum.

Suppose we have a set of individuals, each with experienced utility preferences over some set of social states. Individual preferences over experiences provide us with preference intensities, but such intensities are *not* interpersonally comparable. We can say that individual i likes an apple three times more than a banana and individual j likes an apple two times more than a banana. However, we *cannot* say that i likes an apple more than j likes an apple. To make such comparisons, an individual must be able to put themselves in someone else's shoes. This is not as far of a conceptual leap as it may seem. A complete description of an experience for individual i consists of a complete description of the state of the world over some interval of time. Implicitly, individual i must *exist* in that description and the experience is lived through *her* eyes. An experience in someone *else's* shoes consists of a complete description of the state of the world over an interval of time *and* a specification of the individual whose eyes through which it is lived. Indeed, eating an apple as two different *individuals* is similar to, if not the same as, eating an apple in two different *states of mind*. Hence, eating an apple as i and eating an apple as j are experiences like any other. Preferences over such experiences give rise to *interpersonally comparable* preference intensities.

Let Ω be a finite set of social states and N be a finite set of individuals. Consider a social planner with preferences over $\Omega \times N$, where (ω, i) is interpreted as the (complete) experience of social state ω through the eyes of individual i. Preferences of this form are referred to as *extended sympathy preferences*.[22] Notice that if utility is a representation of individual preferences over Ω, then a social planner can make interpersonal comparisons of utility levels *if and only if* she has extended sympathy preferences.[23] I consider a social planner with extended sympathy preferences over (collections of) the experiences of her citizens $\Omega \times N \times \mathcal{I}$. The statement $(\omega, i, I) \succ (\omega', j, J)$ then means that the experience of state ω through the eyes of individual

[21]Notice that maximizing the sum of arbitrary indexes is not necessarily utilitarianism. For example, a theory that defines individual utilities by *total calories consumed* (and justifies why a social planner should maximize their sum) would likely not be considered a theory of utilitarianism, since most would not consider calories consumed a measure of individual pleasure, etc. Similarly, a theory that justifies some ordering of states and then constructs individual indexes precisely so their sum represents this ordering would not be considered a theory of utilitarianism for the same reason.

[22]See Harsanyi (1955); Arrow (1977).

[23]In particular, $u_i(a) > u_j(a)$ *means* that $(a, i) \succ (a, j)$.

i over the duration I is preferred to the experience of state ω' through the eyes of individual j over the duration J. Formally, the primitive is a tuple $(\mathcal{M}(\Omega\times N\times\mathcal{I}), \succeq)$, where $\succeq$ is a preference relation on $\mathcal{M}(\Omega \times N \times \mathcal{I})$.

An *ethical preference* $\succeq^*$ is a preference relation on social states Ω. The LELO principle is a normative principle which states that what is best for society is what an ethical observer would most prefer if they were to *live every life once*.

The LELO (Live Every Life Once) Principle. A state ω is ethically preferred to another ω' if and only if the complete experience of ω through the eyes of *each* individual is preferred to that of ω'. That is,

$$\omega \succeq^* \omega' \iff \Big\{(\omega, i, [0,1))\Big\}_{i\in N} \succeq \Big\{(\omega', i, [0,1))\Big\}_{i\in N}.$$

Hence, an ethical social planner who respects LELO ranks social states as if she were to experience them through the eyes of each of her citizens. Given this postulate, the following is an immediate implication of Theorem 3.

Corollary 1. *Let $X = \Omega \times N$. Suppose Axioms 3.1, 3.2, 3.3, 3.4, 3.5, and 3.6. An ethical preference $\succeq^*$ on Ω that respects LELO is utilitarian in experienced utilities on $\mathcal{M}(\Omega \times N \times \mathcal{I})$. That is,*

$$\omega \succeq^* \omega' \iff \sum_{i\in N} U(\omega, i, [0,1)) \geq \sum_{i\in N} U(\omega', i, [0,1)),$$

where $\mathcal{U}(\{(\omega, i, [0,1))\}_{i\in N}) = \sum_{i\in N}\int_0^1 v_{\omega i}(t)\ \mathrm{d}t = \sum_{i\in N} U(\omega, i, [0,1))$.

This is a formal theory of utilitarianism. Individual “utilities” are defined as the social planner’s *experienced utility* of the individual’s experience of the state and are unique up to a positive scalar. The LELO principle justifies a particular ranking of social states, and Corollary 1 shows that this ranking is represented by the sum of individual “utilities”. As Sen (1992, p. 12) points out, “every normative theory of social arrangement that has at all stood the test of time seems to demand equality of *something*.” I provide a foundation of utilitarianism from the premise that each citizen’s *experience* should be treated equally.

7 Conclusion

I propose a theory of preference intensity which is built on the idea that the alternatives over which individuals have preferences are experienced over time. Preferences over such experiences yield a measure of preference intensity equal to the integral of hedonic flow. Furthermore, I argue that such a measure, when viewed sympathetically through the eyes of some impartial observer, is precisely the definition of “utility” that utilitarian doctrine fails to specify. In particular, I provide a formal theory of utilitarianism built on the principle that the experience of every citizen should be considered and treated equally.

Appendices

A Measurement Theory

A formal theory of measurement seeks to describe interesting and relevant properties of a given environment numerically.[24] To understand the approach and purpose of measurement theory, it is helpful to consider a simple example—the measurement of "length". Suppose we possess a set of rods. Given any two rods, we may determine which is longer (or that neither is longer). Furthermore, we may form "composite" rods by laying end-to-end any number of individual rods and make the same comparisons between them. This is a formal description of the environment we seek to describe, called a *primitive*, and should be clearly defined and interpretable.[25] We seek to construct a numerical measure, or *representation*, called "length" assigning numbers to rods such that 1) for any two rods A and B, A is longer than B if and only if the length of A is greater than the length of B, and 2) the length of any composite rod is the sum of the length of the individual rods that comprise it. Such a measure is not guaranteed to exist. A *representation theorem* identifies sufficient (and hopefully necessary) conditions, called *axioms*, for its existence, and characterizes the *entire set* of such representations.

A theory of measurement, including a formal specification of the primitive and a representation theorem, serves at least three purposes. First, the axioms identify key properties of the environment that allow for the desired measure to exist. In the case of length, one of the key properties is *monotonicity*, which states that for any rods A, B, and C, if A is longer than B, then the composite rod "A and C" is longer than "B and C". Second, the information captured by the measure is clarified precisely through its implications on the primitive. In particular, what does it *mean* to say the length of rod A is 2 and the length of rod B is 1?[26] Note that if a particular relationship between measurements tells us something about the relationship between primitives, then *all* representations must produce measurements

[24] See Krantz et al. (1971); Suppes et al. (1989); Luce et al. (1990) for an excellent treatment of this material.

[25] That is, we should all agree on what a rod is, what it means for one rod to be longer than another, and what it means to form a composite rod.

[26] The length of A is greater than the length of B means that A is longer than B. The length of A is twice the length of B means the composite rod composed of two Bs is no longer than A, and vice versa.

with this relationship.[27] Third, the constructed measure is usually far more intuitive and efficient in describing the relevant properties than the primitive itself. A single number assigned to each individual rod fully describes an infinite set of comparisons between composite rods.

B Interpretation of Preferences over Experiences

A formal theory of measurement must have a *clearly defined and interpretable* primitive. In this paper, the primitive is a set of *experiences* (or collections of experiences) and a *preference ordering* over those experiences. An experience is a complete description of the state of the world over time. The goal of this section is to clarify what preferences over experiences are intended to capture, i.e. what it *means* to prefer one experience to another.

Different interpretations of preference will result in different interpretations of the axioms (and hence different conclusions about their sensibility) and different interpretations of the utility representation. I have in mind a very particular interpretation, under which I believe the axioms are sensible and the representation meaningful. In fact, under virtually any other interpretation, I would not view the representation as capturing hedonic flow and, likewise, not view the ethical postulate proposed in Section 6 as ethically desirable.

Interpretation of $\succeq$. An individual prefers experience A to B if she would prefer to relive the state of the world described by A to that described by B.

Three points deserve some clarification.

1. The state of the world includes a description of *who* is reliving it and their state of mind at each moment in time. Hence, the state of mind of the individual making the evaluations does not affect the experience.
2. The relived experience does not actually happen—it is only imagined. Hence, nobody else (other than the individual reliving the experience) is affected.
3. The relived experience is forgotten upon completion.[28] Hence, there are no future repercussions to the individual of reliving an experience. The only effect is in the moment.

One implication of these properties is that the timing of evaluation should *not* affect the individual's preference. Suppose an individual considered her actions on a particular day to be morally right *at the time*, but now views them as morally wrong. Consider the experience of reliving that day. Since the experience is only

[27] Since the length of A being twice that of B captures information about the primitive (see previous footnote), all representations must share this fact.

[28] Alternatively, the experience is relived at the end of the individual's life, just after death.

imagined (so it affects no one else) and she relives it in her *original* state of mind, her present moral views are not relevant.

A second implication of these properties is that there is nothing to be gained or lost from reliving an experience other than the experience itself. For instance, an individual cannot relive a day of intense studying in order to better learn the material in her actual life.

Temporal monotonicity is the key axiom giving rise to the representation. Point 1 is clearly necessary for such an axiom to be justifiable. If the experience did not include the individual's state of mind, there would be no difference between the experience of "the first bite of an apple" and "the last bite of an apple" (assuming everything else about these bites are the same). That is, the only difference between the first bite and the last is precisely the individual's state of mind.

Point 3 is also necessary for the axiom to be justifiable. Suppose Points 1 and 2, but not 3. Consider the experience of learning a password to some valuable safe. Learning the first four digits is useless later in one's actual life, as is learning the last four digits. But learning all eight digits is very useful.

Point 2, on the other hand, is not necessary for the sensibility of temporal monotonicity. It is, however, necessary to interpret the representation as "hedonic flow". Suppose Points 1 and 3, but not 2. Then the individual evaluates experiences given that *everyone* will relive it. Hence, she takes into account her moral views at the time of evaluation, including any altruistic value she places on others. This drastically changes the interpretation of her preference (though it should still satisfy temporal monotonicity) and the resulting representation.

C Proofs

C.1 Proof of Theorem 1

If preferences are trivial (all experiences are indifferent), then $v(t) = 0$ for all t represents preferences and Theorem 1 follows immediately. Henceforth, I assume preferences are *non-trivial*. I use $(x, y) \in [0, 1]^2$ to represent $[x, y) \in \mathcal{I}$ and, where there is no confusion, 0 to denote the empty experience $\emptyset$.

Lemma 1 provides us with a helpful trick to switch the "inner coordinates" of a pair of intervals (for which one is not contained in the another).

Lemma 1. *Suppose Axioms 1.1 and 1.2. Then for any* $0 \le t_1 \le t_2 \le t_3 \le t_4 \le 1$,

$$(t_1, t_2) \succeq (t_3, t_4) \iff (t_1, t_3) \succeq (t_2, t_4).$$

Proof. ($\Rightarrow$) By Axiom 1.2, $(t_1, t_2) \succeq (t_3, t_4)$ and $(t_2, t_3) \sim (t_2, t_3)$ implies $(t_1, t_3) \succeq (t_2, t_4)$. ($\Leftarrow$) Suppose by contradiction that $(t_1, t_3) \succeq (t_2, t_4)$ and $(t_3, t_4) \succ (t_1, t_2)$.

By Axiom 1.2, $(t_2, t_3) \sim (t_2, t_3)$ and $(t_3, t_4) \succ (t_1, t_2)$ implies $(t_2, t_4) \succ (t_1, t_3)$, a contradiction. ■

Lemma 2 says that each vertical (horizontal) line in the triangle has the same preference ordering as every other vertical (horizontal) line, and that every vertical ordering is the opposite of its corresponding horizontal ordering.

Lemma 2. *Suppose Axioms 1.1 and 1.2. Then for any* $0 \leq x \leq y \leq y' \leq z \leq 1$,

$$(x, y) \succeq (x, y') \iff (y', z) \succeq (y, z).$$

Proof. By Lemma 1, $(x, y) \succeq (x, y')$ if and only if $(x, x) \succeq (y, y')$ if and only if $(z, z) \succeq (y, y')$ if and only if $(y', z) \succeq (y, z)$. Recall that (x, x) and (z, z) represent the same empty interval. ■

Define

$$\hat{V}(z) = \int_0^z \hat{v}(t)\ \mathrm{d}t,$$

and

$$\hat{U}(x, y) = \int_x^y \hat{v}(t)\ \mathrm{d}t = \hat{V}(y) - \hat{V}(x).$$

This will serve as a conjecture for the representation U, which we will construct later. Lemma 3 says that, at any point (x, y) for which $\hat{v}(x) \neq 0$ or $\hat{v}(y) \neq 0$, the slope of the indifference curve through (x, y) generated by $\hat{U}(x, y) = \hat{V}(y) - \hat{V}(x)$ agrees with the slope of *some* indifference curve through (x, y) generated by preferences $\succeq$.

Lemma 3. *Suppose Axioms 1, 2, 3, 4 and that* $\succeq$ *are non-trivial. If* $\hat{v}(y) \neq 0$, *then there exists* $f_{xy} \in F_{xy}$ *such that* $f'_{xy}(x) = \hat{v}(x)/\hat{v}(y)$. *If* $\hat{v}(x) \neq 0$, *then there exists* $g_{xy} \in G_{xy}$ *such that* $g'_{xy}(y) = \hat{v}(y)/\hat{v}(x)$.

Proof. **Case 1:** $0 \leq x \leq y < \tau \leq 1$

Part I: Suppose $\hat{v}(y) = \phi'_{y\tau}(y) \neq 0$. By the right-inverse function theorem,[29] there exists a right-inverse $\phi^{-1}_{y\tau}(\cdot)$ of $\phi_{y\tau}(\cdot)$ defined on a neighborhood of τ such that $\phi'^{-1}_{y\tau}(\tau) = 1/\phi'_{y\tau}(y)$. Then there exists $\delta > 0$ such that for all $\hat{x} \in N(x, \delta)$, $(x, \tau) \sim (\hat{x}, \phi_{x\tau}(\hat{x}))$ and $(y, \tau) \sim (\phi^{-1}_{y\tau}(\phi_{x\tau}(\hat{x})), \phi_{x\tau}(\hat{x}))$. By Axiom 1.2, $(x, y) \sim (\hat{x}, \phi^{-1}_{y\tau}(\phi_{x\tau}(\hat{x}))$. Let $f_{xy}(\hat{x}) \equiv \phi^{-1}_{y\tau}(\phi_{x\tau}(\hat{x}))$. Then $f_{xy} \in F_{xy}$ and $f'_{xy}(x) = \phi'_{x\tau}(x)/\phi'_{y\tau}(y) = \hat{v}(x)/\hat{v}(y)$.

Part II: Suppose $\hat{v}(x) = \phi'_{x\tau}(x) \neq 0$. By the right-inverse function theorem, there exists a right-inverse $\phi^{-1}_{x\tau}(\cdot)$ defined on a neighborhood of τ such that $\phi'^{-1}_{x\tau}(\tau) =$

[29] See Radulescu and Radulescu (1989).

$1/\phi'_{x\tau}(x)$. Then there exists $\delta > 0$ such that for all $\hat{y} \in N(y, \delta)$, $(y, \tau) \sim (\hat{y}, \phi_{y\tau}(\hat{y}))$ and $(x, \tau) \sim (\phi_{x\tau}^{-1}(\phi_{y\tau}(\hat{y})), \phi_{y\tau}(\hat{y}))$. By Axiom 1.2, $(x, y) \sim (\phi_{x\tau}^{-1}(\phi_{y\tau}(\hat{y})), \hat{y})$. Let $g_{xy}(\hat{y}) \equiv \phi_{x\tau}^{-1}(\phi_{y\tau}(\hat{y}))$. Then $g_{xy} \in G_{xy}$ and $g'_{xy}(y) = \phi'_{y\tau}(y)/\phi'_{x\tau}(x) = \hat{v}(y)/\hat{v}(x)$.

Case 2: $0 \leq x < \tau \leq y \leq 1$

Part I: Suppose $\hat{v}(y) = \gamma'_{\tau y}(y) \neq 0$. By the right-inverse function theorem, there exists a right-inverse $\gamma_{\tau y}^{-1}(\cdot)$ defined on a neighborhood of τ such that $\gamma'^{-1}_{\tau y}(\tau) = 1/\gamma'_{\tau y}(y)$. Then there exists $\delta > 0$ such that for all $\hat{x} \in N(x, \delta)$, $(x, \tau) \sim (\hat{x}, \phi_{x\tau}(\hat{x}))$ and $(\tau, y) \sim (\phi_{x\tau}(\hat{x}), \gamma_{\tau y}^{-1}(\phi_{x\tau}(\hat{x})))$. By Axiom 1.2, $(x, y) \sim (\hat{x}, \gamma_{\tau y}^{-1}(\phi_{x\tau}(\hat{x})))$. Let $f_{xy}(\hat{x}) \equiv \gamma_{\tau y}^{-1}(\phi_{x\tau}(\hat{x}))$. Then $f_{xy} \in F_{xy}$ and $f'_{xy}(x) = \phi'_{x\tau}(x)/\gamma'_{\tau y}(y) = \hat{v}(x)/\hat{v}(y)$.

Part II: Suppose $\hat{v}(x) = \phi'_{x\tau}(x) \neq 0$. By the right-inverse function theorem, there exists a right-inverse $\phi_{x\tau}^{-1}(\cdot)$ defined on a neighborhood of τ such that $\phi'^{-1}_{x\tau}(\tau) = 1/\phi'_{x\tau}(x)$. Then there exists $\delta > 0$ such that for all $\hat{y} \in N(y, \delta)$, $(x, \tau) \sim (\phi_{x\tau}^{-1}(\gamma_{\tau y}(\hat{y})), \gamma_{\tau y}(\hat{y}))$ and $(\tau, y) \sim (\gamma_{\tau y}(\hat{y}), \hat{y})$. By Axiom 1.2, $(x, y) \sim (\phi_{x\tau}^{-1}(\gamma_{\tau y}(\hat{y})), \hat{y})$. Let $g_{xy}(\hat{y}) \equiv \phi_{x\tau}^{-1}(\gamma_{\tau y}(\hat{y}))$. Then $g_{xy} \in G_{xy}$ and $g'_{xy}(y) = \gamma'_{\tau y}(y)/\phi'_{x\tau}(x) = \hat{v}(y)/\hat{v}(x)$.

Case 3: $0 \leq \tau \leq x \leq y \leq 1$

Part I: Suppose $\hat{v}(y) = \gamma'_{\tau y}(y) \neq 0$. By the right-inverse function theorem, there exists a right-inverse $\gamma_{\tau y}^{-1}(\cdot)$ defined on a neighborhood of τ such that $\gamma'^{-1}_{\tau y}(\tau) = 1/\gamma'_{\tau y}(y)$. Then there exists $\delta > 0$ such that for all $\hat{x} \in N(x, \delta)$, $(\tau, x) \sim (\gamma_{\tau x}(\hat{x}), \hat{x})$ and $(\tau, y) \sim (\gamma_{\tau x}(\hat{x}), \gamma_{\tau y}^{-1}(\gamma_{\tau x}(\hat{x})))$. By Axiom 1.2, $(x, y) \sim (\hat{x}, \gamma_{\tau y}^{-1}(\gamma_{\tau x}(\hat{x})))$. Let $f_{xy}(\hat{x}) \equiv \gamma_{\tau y}^{-1}(\gamma_{\tau x}(\hat{x}))$. Then $f_{xy} \in F_{xy}$ and $f'_{xy}(x) = \gamma'_{\tau x}(x)/\gamma'_{\tau y}(y) = \hat{v}(x)/\hat{v}(y)$.

Part II: Suppose $\hat{v}(x) = \gamma'_{\tau x}(x) \neq 0$. By the right-inverse function theorem, there exists a right-inverse $\gamma_{\tau x}^{-1}(\cdot)$ defined on a neighborhood of τ such that $\gamma'^{-1}_{\tau x}(\tau) = 1/\gamma'_{\tau x}(x)$. Then there exists $\delta > 0$ such that for all $\hat{y} \in N(y, \delta)$, $(\tau, y) \sim (\gamma_{\tau y}(\hat{y}), \hat{y})$ and $(\tau, x) \sim (\gamma_{\tau y}(\hat{y}), \gamma_{\tau x}^{-1}(\gamma_{\tau y}(\hat{y})))$. By Axiom 1.2, $(x, y) \sim (\gamma_{\tau x}^{-1}(\gamma_{\tau y}(\hat{y})), \hat{y})$. Let $g_{xy}(\hat{y}) \equiv \gamma_{\tau x}^{-1}(\gamma_{\tau y}(\hat{y}))$. Then $g_{xy} \in G_{xy}$ and $g'_{xy}(y) = \gamma'_{\tau y}(y)/\gamma'_{\tau x}(x) = \hat{v}(y)/\hat{v}(x)$. ■

Lemma 4 says that if two points are on the same indifference curve generated by $\hat{U}(x, y) = \hat{V}(y) - \hat{V}(x)$, then they are indifferent according to $\succeq$.

Lemma 4. *At any point (x, y), small changes in x and y that hold $\hat{V}(y) - \hat{V}(x)$ constant do not change the preference.*

Proof. If preferences are trivial, this holds by definition. Suppose preferences are non-trivial. Consider a point (x, y) at which $\hat{v}(y) \neq 0$. Small changes with $\mathrm{d}y/\mathrm{d}x = \hat{v}(x)/\hat{v}(y)$ hold $\hat{V}(y) - \hat{V}(x)$ constant, and by Lemma 3, do not change the preference. Consider a point (x, y) at which $\hat{v}(x) \neq 0$. Small changes with $\mathrm{d}y/\mathrm{d}x = \hat{v}(y)/\hat{v}(x)$ hold $\hat{V}(y) - \hat{V}(x)$ constant, and by Lemma 3, do not change the preference. In other words, the indifference map generated by $\hat{U}(x, y) = \hat{V}(y) - \hat{V}(x)$ and the indifference map generated by preferences $\succeq$ agree for all points at which $\hat{v}(x) \neq 0$ or $\hat{v}(y) \neq 0$.

Now, consider a point (x, y) at which $\hat{v}(x) = \hat{v}(y) = 0$. For any $z \in [0,1]$ such that $\hat{v}(z) = 0$, let $[\underline{z}, \overline{z}]$ be the largest interval around z for which $\hat{v}(t) = 0$ for all $t \in [\underline{z}, \overline{z}]$.

We would first like to show that for any z such that $\hat{v}(z) = 0$, $(\underline{z}, z') \sim 0$ for all $z' \in [\underline{z}, \overline{z}]$. If $z < \tau$, then $\underline{z} \leq \overline{z} < \tau$ by Axiom 1.4, $(\underline{z}, \tau) \sim (z', \tau)$ (since on the same horizontal indifference curve), and $(\underline{z}, z') \sim (z', z') \sim 0$ by Lemma 2. If $z > \tau$, then $\tau < \underline{z} \leq \overline{z}$ by Axiom 1.4, $(\tau, \underline{z}) \sim (\tau, z')$ (since on the same vertical indifference curve), and $0 \sim (\underline{z}, \underline{z}) \sim (\underline{z}, z')$ by Lemma 2.

We would now like to show that $(\underline{x}, \underline{y}) \sim (x', y')$ for any $x' \in [\underline{x}, \overline{x}]$ and $y' \in [\underline{y}, \overline{y}]$ with $x' \leq y'$. From the previous result, $(\underline{x}, x') \sim 0$ and $(\underline{y}, y') \sim 0$. By Axiom 1.2, $(\underline{x}, \underline{y}) \sim (x', y')$.

For any (x, y) with $\hat{v}(x) = \hat{v}(y) = 0$, there exists an indifference curve for which $\hat{U}(x, y)$ and $\succeq$ agree that gets arbitrarily close to $(\underline{x}, \underline{y})$ and an indifference curve for which $\hat{U}(x, y)$ and $\succeq$ agree that gets arbitrarily close to $(\overline{x}, \overline{y})$. By Axiom 1.3, $(\underline{x}, \underline{y})$ and $(\overline{x}, \overline{y})$ are part of such indifference curves, respectively, and by the previous result, all points in the rectangle connecting $(\underline{x}, \underline{y})$, $(\overline{x}, \underline{y})$, $(\overline{x}, \overline{y})$, and $(\underline{x}, \overline{y})$ are indifferent. ■

The following definition says that a point (x, y) is *increasing to the left* (inc-left) if all points directly to the left within some neighborhood are strictly preferred to it (and likewise for the other cases).

Definition 4.1. (x, y) is *inc(dec)-left* if $\exists \delta > 0$, $\forall \varepsilon \in (0, \delta)$, $(x - \varepsilon, y) \succ(\prec) (x, y)$, *inc(dec)-right* if $\exists \delta > 0$, $\forall \varepsilon \in (0, \delta)$, $(x + \varepsilon, y) \succ(\prec) (x, y)$, *inc(dec)-down* if $\exists \delta > 0$, $\forall \varepsilon \in (0, \delta)$, $(x, y - \varepsilon) \succ(\prec) (x, y)$, and *inc(dec)-up* if $\exists \delta > 0$, $\forall \varepsilon \in (0, \delta)$, $(x, y + \varepsilon) \succ(\prec) (x, y)$.

Let me refer to the line connecting $(0, \tau)$, (τ, τ), and $(\tau, 1)$ as the τ-ray. Lemma 5 says that there must exist some point (x^*, τ) on the horizontal portion of the τ-ray that is either increasing upward or decreasing upward *and* either increasing downward or decreasing downward. (This is also true on the vertical portion, but we will not need this fact.)

Lemma 5. *Suppose Axioms 1.1, 1.2, 1.3, and 1.4. Then there exists* $0 < x^* < \tau$ *such that* (x^*, τ) *is inc-up or dec-up* and *inc-down or dec-down.*

Proof. By Axiom 1.4, there exists $0 < x < \tau$ such that $\hat{v}(x) \neq 0$. Suppose by contradiction that (x, τ) is neither inc-up *nor* dec-up. Then by Axiom 1.3, for any $\delta > 0$, there exists $\varepsilon \in (0, \delta)$ such that $(x, \tau + \varepsilon) \sim (x, \tau)$. For sufficiently small ε, there exists $h \in \mathbb{R}$ such that $\tau + \varepsilon = \phi_{x\tau}(x + h)$.[30] By Lemma 2, $(x + h, \phi_{x\tau}(x + \varepsilon)) = (x + h, \tau + \varepsilon) \sim (x + h, \tau)$. But this is a contradiction, since $\phi_{x\tau}(x + \varepsilon)$ is an indifference

[30] Note that h may be negative.

curve formed with points *closest* to the horizontal τ. The same technique is used to show (x^*, τ) must be inc-down or dec-down. ■

Lemma 6 says that a point (x, y) on the τ-ray is increasing to the left if and only if it is decreasing downward, if $\hat{v}(t)$ is positive, or upward, if $\hat{v}(t)$ is negative. Similarly for the other cases.

Lemma 6. *Suppose Axioms 1.1, 1.2, and 1.4 and let* $0 \le x < \tau < y \le 1$. *If* $\hat{v}(x) > 0$, *then* (x, τ) *is inc(dec)-right if and only if it is dec(inc)-up, and* (x, τ) *is inc(dec)-left if and only if it is dec(inc)-down. If* $\hat{v}(x) < 0$, *then* (x, τ) *is inc(dec)-right if and only if it is dec(inc)-down, and* (x, τ) *is inc(dec)-left if and only if it is dec(inc)-up. The same is true for* (τ, y) *given* $\hat{v}(y) > 0$ *and* $\hat{v}(y) < 0$, *respectively.*

Proof. Suppose $\hat{v}(x) = \phi'_{x\tau}(x) > 0$. Then $\exists \delta > 0$, $\forall \varepsilon \in (0, \delta)$, $(x, \tau) \sim (x + \varepsilon, \phi_{x\tau}(x + \varepsilon))$ and $\phi_{x\tau}(x + \varepsilon) > \tau$. By Lemma 2, $(x + \varepsilon, \tau) \succ(\prec) (x, \tau)$ if and only if $(x, \tau) \sim (x + \varepsilon, \phi_{x\tau}(x + \varepsilon)) \succ(\prec) (x, \phi_{x\tau}(x + \varepsilon))$. Hence, (x, τ) is inc(dec)-right if and only if it is dec(inc)-up. The other three cases may be shown in the same manner. Now suppose $\hat{v}(y) = \gamma'_{\tau y}(y) > 0$. Then $\exists \delta > 0$, $\forall \varepsilon \in (0, \delta)$, $(\tau, y) \sim (\gamma_{\tau y}(y + \varepsilon), y + \varepsilon)$ and $\gamma_{\tau y}(x + \varepsilon) > \tau$. By Lemma 2, $(\tau, y + \varepsilon) \succ(\prec) (\tau, y)$ if and only if $(\tau, y) \sim (\gamma_{\tau y}(y + \varepsilon), y + \varepsilon) \succ(\prec) (\gamma_{\tau y}(y + \varepsilon), y)$. Hence, (τ, y) is inc(dec)-up if and only if it is dec(inc)-right. The other three cases may be shown in the same manner. ■

Lemma 7 says that if a single point on the horizontal portion of the τ-ray is increasing upward, then *all* points on the horizontal portion are increasing upward and *all* points on the vertical portion are decreasing to the right. Similarly for the other cases.

Lemma 7. *Suppose Axioms 1.1 and 1.2 and let* $0 \le x < \tau < y \le 1$. *Then* (x, τ) *is inc(dec)-up if and only if* (τ, y) *is dec(inc)-right, and* (x, τ) *is inc(dec)-down if and only if* (τ, y) *is dec(inc)-left.*

Proof. By Lemma 2, $\exists \delta > 0$, $\forall \varepsilon \in (0, \delta)$, $(x, \tau + \varepsilon) \succ(\prec) (x, \tau)$ if and only if $(\tau, \tau + \varepsilon) \succ(\prec) (\tau, \tau)$ if and only if $(\tau, \tau + \varepsilon) \succ(\prec) (\tau + \varepsilon, \tau + \varepsilon)$ if and only if, for $y > \tau + \varepsilon$, $(\tau, y) \succ(\prec) (\tau + \varepsilon, y)$. Hence, (x, y) is inc(dec)-up if and only if it is dec(inc)-right. The other case may be shown in the same manner. ■

We are now ready to construct the representation U. Define

$$v(t) = \begin{cases} \hat{v}(t) & \text{if } (x^*, \tau) \text{ is inc-up} \\ -\hat{v}(t) & \text{if } (x^*, \tau) \text{ is dec-up} \end{cases},$$

$$V(z) = \int_0^z v(t)\ \mathrm{d}t,$$

and

$$U(x, y) = \int_x^y v(t)\ \mathrm{d}t = V(y) - V(x).$$

Lemma 8 says that small changes in x or y that increase $V(y) - V(x)$ increase the preference, and small changes in x or y that decrease $V(y) - V(x)$ decrease the preference.

Lemma 8. *Suppose Axioms 1.1, 1.2, 1.3, and 1.4. At any point (x, y), small changes in x that increase V decrease the preference and small changes in x that decrease V increase the preference, while small changes in y that increase V increase the preference and small changes in y that decrease V decrease the preference.*

Proof. First, we would like to show that (x^*, τ) is inc(dec)-up if and only if it is dec(inc)-down. By Lemma 5, (x^*, τ) is either inc-up or dec-up *and* either inc-down or dec-down. Suppose $\hat{v}(x^*) > 0$. By Lemma 6, if (x^*, τ) is inc(dec)-up it is also dec(inc)-right, and if (x^*, τ) is inc(dec)-down it is also dec(inc)-left. Suppose by contradiction that (x^*, τ) is inc(dec)-up and inc(dec)-down. Then by Lemma 7 *all* points on the horizontal portion of the τ-ray are inc(dec)-up and inc(dec)-down, and by Lemma 6 these points are also dec(inc)-right and dec(inc)-left. But all points in an interval cannot be dec(inc)-right and dec(inc)-left,[31] so we have a contradiction. The same can be shown in the case of $\hat{v}(x^*) < 0$.

By Lemma 6, for any $x < y$, $v(x) > 0$ implies (x, τ) is inc-left and dec-right, $v(x) < 0$ implies (x, τ) is dec-left and inc-right, $v(y) > 0$ implies (τ, y) is dec-down and inc-up, and $v(y) < 0$ implies (τ, y) is inc-down and dec-up. By Lemma 2, this holds for any (x, y). The result follows. ■

Definition 4.2. A function f is non-decreasing (non-increasing) to the right at x if there exists $\delta > 0$ such that f is non-decreasing (non-increasing) on $[x, x + \delta]$, and is non-decreasing (non-increasing) to the left at x if there exists $\delta > 0$ such that f is non-decreasing (non-increasing) on $[x - \delta, x]$. Similarly for (strictly) increasing/decreasing to the right/left.

Lemma 9 says that comparisons to 0 under the representation U imply the corresponding comparison to 0 under $\succeq$.

Lemma 9. *Suppose Axioms 1.1, 1.2, 1.3, and 1.4. Then $U(I) > 0 \implies I \succ 0$, $U(I) = 0 \implies I \sim 0$, and $U(I) < 0 \implies I \prec 0$.*

Proof. Let $I = (x, y)$ so that $U(I) = V(y) - V(x)$.

Part I: $U(I) = 0 \implies I \sim 0$

[31] A function cannot have a local max(min) at every point in an interval.

("Squeeze and Partition") Suppose $V(y) = V(x)$. As long as V is non-decreasing (non-increasing) to the right at x and non-increasing (non-decreasing) to the left at y, squeeze (x, y) inward[32] until we reach a point (x', y') with $V(y') = V(x') = k$ where (1) $x' = y'$, or (2) V is decreasing to the right at x' and left at y' or increasing to the right at x' and left at y'. If (1), then $(x, y) \sim (x', y') \sim 0$, and we are done. If (2), then there exists $t \in (x', y')$ such that $V(t) = k$. Repeat this process for (x, t) and (t, y). In (2), $V' = v$ must cross zero at least twice in (x', y'). Since v crosses zero finitely many times, this process must terminate. By Axiom 1.2, $(x, y) \sim 0$.

Part II: $U(I) >(<)\, 0 \implies I \succ(\prec)\, 0$

("Slide and Jump") Suppose $V(y) >(<) V(x)$. As long as V is non-decreasing (non-increasing) to the right at x, slide x right holding y constant, decreasing (increasing) the preference (by Lemma 8), until we reach an x' such that (1) $V(x') = V(y)$, or (2) V is decreasing (increasing) to the right at x. If (1), then $(x, y) \succ(\prec)\, (x', y) \sim 0$ (by Part I), and we are done. If (2), then there exists $x'' \in (x', y)$ such that $V(x'') = V(x')$. By Part I $(x', x'') \sim 0$ and by Axiom 1.2 $(x, y) \succ(\prec)\, (x', y) \sim (x'', y)$. Repeat this process for (x'', y). In (2), $V' = v$ must cross zero at least once in (x', y). Since v crosses zero finitely many times, this process must terminate. Hence, $(x, y) \succ(\prec)\, 0$. ∎

Suppose $V(x) {\geq}({\leq}) V(y)$. Lemma 10 says that for any $\hat{x} \leq y$ with $V(\hat{x}) {\geq}({\leq}) V(x)$, we can find a $\hat{y}$ such that $(\hat{x}, \hat{y}) \sim (x, y)$. Similarly, for any $\hat{y} \geq x$ with $V(\hat{y}) \geq(\leq) V(y)$, we can find a $\hat{x}$ such that $(\hat{x}, \hat{y}) \sim (x, y)$.

Lemma 10. *Suppose Axioms 1.1, 1.2, 1.3, and 1.4. Let $x \leq y$. Suppose $V(x) {\geq}({\leq}) V(y)$. For any $\hat{x} \in [0, y]$ with $V(\hat{x}) {\geq}({\leq}) V(x)$, there exists $\hat{y} \in [\hat{x}, y]$ such that $V(\hat{y}) - V(\hat{x}) = V(y) - V(x)$ and $(\hat{x}, \hat{y}) \sim (x, y)$. For any $\hat{y} \in [x, 1]$ with $V(\hat{y}) \geq(\leq) V(y)$, there exists $\hat{x} \in [x, \hat{y}]$ such that $V(\hat{y}) - V(\hat{x}) = V(y) - V(x)$ and $(\hat{x}, \hat{y}) \sim (x, y)$.*

Proof. Let $x \leq y$ and $V(x) \geq(\leq) V(y)$.

Suppose $\hat{x} \in [0, x]$ and $V(\hat{x}) \geq(\leq) V(x)$. If V is non-increasing (non-decreasing) to the left at x and y, slide (x, y) to the left[33] until (1) $V(x') = V(\hat{x})$, or (2) V is increasing (decreasing) to the left at x' or at y'. If (1), then $(\hat{x}, x') \sim 0$ by Lemma 9 and $(\hat{x}, y') \sim (x', y') \sim (x, y)$ by Axiom 1.2, so we are done. If (2), then proceed as follows. If V is increasing (decreasing) to the left at x', then there exists $x'' \in (\hat{x}, x')$ such that $V(x'') = V(x')$ and by Lemma 9, $(x'', x') \sim 0$. If not, let $x'' = x'$. If V is increasing (decreasing) to the left at y', then there exists $y'' \in (x', y')$ such that $V(y'') = V(y')$ and by Lemma 9, $(y'', y') \sim 0$. If not, let $y'' = y'$. By Axiom 1.2,

[32]By this I mean increase x and decrease y holding $U(x, y) = V(y) - V(x)$ constant, which by Lemma 4, keeps the individual indifferent.

[33]By this I mean increase x and increase y holding $U(x, y) = V(y) - V(x)$ constant, which by Lemma 4, keeps the individual indifferent.

$(x'', y'') \sim (x', y') \sim (x, y)$. Repeat this process for (x'', y''). In (2), $V' = v$ must cross zero at least once within $(\hat{x}, x')$ or (x', y'). Since v crosses zero finitely many times, this process must terminate, and the result follows.

Suppose $\hat{x} \in (x, y]$ and $V(\hat{x}) \geq (\leq)\, V(x)$. If V is non-decreasing (non-increasing) to the right at x and non-increasing (non-decreasing) to the left at y, squeeze (x, y) inward[34] until (1) $V(x') = V(\hat{x})$, or (2) V is decreasing (increasing) to the right at x' or increasing (decreasing) to the left at y'. If (1), then $(x', \hat{x}) \sim 0$ by Lemma 9 and $(\hat{x}, y') \sim (x', y') \sim (x, y)$ by Axiom 1.2, so we are done. If (2), then proceed as follows. If V is decreasing (increasing) to the right at x', then there exists $x'' \in (x', \hat{x})$ such that $V(x'') = V(x')$ and by Lemma 9, $(x', x'') \sim 0$. If not, let $x'' = x'$. If V is increasing (decreasing) to the left at y', then there exists $y'' \in (\hat{x}, y')$ such that $V(y'') = V(y')$ and by Lemma 9, $(y'', y') \sim 0$. If not, let $y'' = y'$. By Axiom 1.2, $(x'', y'') \sim (x', y') \sim (x, y)$. Repeat this process for (x'', y''). In (2), v must cross zero at least once within $(x', \hat{x})$ or $(\hat{x}, y')$. Since v crosses zero finitely many times, this process must terminate, and the result follows.

A similar proof applies for the latter half of the lemma. ■

Finally, Lemma 11 says that U represents $\succeq$.

Lemma 11. *Suppose Axioms 1.1, 1.2, 1.3, and 1.4. Then* $I \succeq J \iff U(I) \geq U(J)$.

Proof. We seek to show that for any I and J, $U(I) > (=)(<) U(J) \implies I \succ (\sim)(\prec) J$. Let $I = (x, y)$ and $J = (a, b)$. If $U(I)$ and $U(J)$ have opposite signs or at least one is zero, then the result follows immediately by Lemma 9. Suppose I and J are non-zero with the same sign, so that $U(I), U(J) > 0$ or $U(I), U(J) < 0$. We will prove the result for $U(I), U(J) > 0$. The proof for $U(I), U(J) < 0$ follows in a similar fashion.

Case 1 (Disjoint Intervals): $x \leq y \leq a \leq b$

1. $V(b) \geq V(y)$
   - By Lemma 10, there exists $a' \in [x, b]$ such that $V(b) - V(a') = V(y) - V(x)$ and $(a', b) \sim (x, y)$. By Lemma 1 and 9: if $a \leq a'$, $U(I) > (=)(<) U(J)$ implies $V(a') - V(a) < (=)(>)\, 0$ implies $(a, a') \prec (\sim)(\succ)\, 0$ implies $(a', b) \succ (\sim)(\prec)\, (a, b)$; and if $a' < a$, $U(I) > (=)(<) U(J)$ implies $V(a) - V(a') > (=)(<)\, 0$ implies $(a', a) \succ (\sim)(\prec)\, 0$ implies $(a', b) \succ (\sim)(\prec)\, (a, b)$.
2. $V(y) > V(b)$

[34] See Footnote 32 for definition.

- By Lemma 1, $(x, y) \succeq (a, b) \iff (x, a) \succeq (y, b)$. By Lemma 10, there exists $a' \in [y, a]$ such that $V(a) - V(x) = V(a') - V(y)$ and $(x, a) \sim (y, a')$. Similarly, there exists $y' \in [y, b]$ such that $V(b) - V(y) = V(a) - V(y')$ and $(y, b) \sim (y', a)$. By Lemma 1, $(y, a') \succeq (y', a) \iff (y, y') \succeq (a', a)$. Notice that $y \leq y' \leq a' \leq a$, $V(y') - V(y) = V(a) - V(b) < 0$, and $V(a) - V(a') = V(x) - V(y) < 0$. Hence, we are in Case 1-1 with $U(I), U(J) < 0$, which follows in the same manner as Case 1-1 with $U(I), U(J) > 0$.

Case 2 (Overlapping Intervals): $x \leq a \leq y \leq b$

By Lemma 1, $(x, y) \succeq (a, b) \iff (x, a) \succeq (y, b)$, bringing us to Case 1.

Case 3 (Containing Intervals): $a \leq x \leq y \leq b$

1. $V(a) < V(b) \leq V(x) < V(y)$
   - By Lemma 10, there exists $b' \in [a, y]$ such that $V(b') - V(a) = V(y) - V(x)$ and $(a, b') \sim (x, y)$. By Lemma 1 and 9, $U(I) > (=)(<) U(J)$ implies $V(b) - V(b') <(=)(>) 0$ implies $(b', b) \prec(\sim)(\succ) 0$ implies $(a, b') \succ(\sim)(\prec) (a, b)$.
2. $V(x) < V(y) \leq V(a) < V(b)$
   - By Lemma 10, there exists $a' \in [x, b]$ such that $V(b) - V(a') = V(y) - V(x)$ and $(a', b) \sim (x, y)$. By Lemma 1 and 9, $U(I) >(=)(<) U(J)$ implies $V(a') - V(a) <(=)(>) 0$ implies $(a, a') \prec(\sim)(\succ) 0$ implies $(a', b) \succ(\sim)(\prec) (a, b)$.
3. $V(a) \leq V(x) < V(y) \leq V(b)$
   - Note that only $U(I) < U(J)$ and $U(I) = U(J)$ are possible here. By Lemma 9, $V(x) - V(a) > (=) 0$ implies $(a, x) \succ (\sim) 0$ and $V(b) - V(y) > (=) 0$ implies $(y, b) \succ(\sim) 0$. $U(I) <(=) U(J)$ implies at least one $>$ (both $=$), and hence $(x, y) \prec(\sim) (a, b)$ by Axiom 1.2.[35]
4. $V(x) \leq V(a) < V(b) \leq V(y)$
   - Note that only $U(I) > U(J)$ and $U(I) = U(J)$ are possible here. By Lemma 9, $V(x) - V(a) < (=) 0$ implies $(a, x) \prec (\sim) 0$ and $V(b) - V(y) < (=) 0$ implies $(y, b) \prec(\sim) 0$. $U(I) >(=) U(J)$ implies at least one $<$ (both $=$), and hence $(x, y) \succ(\sim) (a, b)$ by Axiom 1.2.

$\blacksquare$

We are now ready to prove Theorem 1.

[35] Note that we could apply either of the previous two arguments here as well.

Theorem 1. *Axioms 1.1, 1.2, 1.3, and 1.4 hold if and only if there exists an instantaneous utility function $v : [0,1] \to \mathbb{R}$ that is continuous and crosses zero finitely many times, such that $\succeq$ is represented by the experienced utility function $U : \mathcal{I} \to \mathbb{R}$ given by*

$$U(I) = \int_I v(t) \, \mathrm{d}t.$$

Moreover, $\tilde{U}(I) = \int_I \tilde{v}(t) \, \mathrm{d}t$ is another representation if and only if $\tilde{v} = \alpha v$ for some $\alpha > 0$.

Proof. By Lemma 11, Axioms 1.1, 1.2, 1.3, and 1.4 imply U represents $\succeq$. That U represents $\succeq$ implies Axioms 1.1, 1.2, 1.3, and 1.4 is straightforward and left to the reader. Lastly, we seek to show that if $U(I) = \int_I v(t) \, \mathrm{d}t$ is an experienced utility representation of $\succeq$, then $\tilde{U}(I) = \int_I \tilde{v}(t) \, \mathrm{d}t$ is also an experienced utility representation of $\succeq$ if and only if $\tilde{v} = \alpha v$ for some $\alpha > 0$. If preferences are trivial, this is immediate. Suppose preferences are non-trivial. The proof of $\Leftarrow$ is straightforward and left to the reader. To show $\Rightarrow$, recall that for any $x < \tau$ the slope of the indifference curve at (x, τ) is $v(x)/v(\tau)$ and for any $y > \tau$ the inverse slope of the indifference curve at (τ, y) is $v(y)/v(\tau)$. To match these slopes, $\hat{v}(\tau) \neq 0$ and $\hat{v}(t)/\hat{v}(\tau) = v(t)/v(\tau)$ for all $t \in [0,1]$. Hence, $\hat{v}(t) = v(t) \cdot \hat{v}(\tau)/v(\tau)$ for all $t \in [0,1]$. But $\tilde{U}(I) = \int_I \tilde{v}(t) \, \mathrm{d}t$ only represents preferences $\succeq$ if $\alpha = \hat{v}(\tau)/v(\tau) > 0$ (if $\alpha < 0$ the preferences would be reversed), completing the proof. ■

C.2 Proof of Theorem 2

By Axiom 2.5, all empty experiences are indifferent, so when there is no confusion we may denote any such experience by 0.

Definition 4.1. A set of experiences $\{(x, I_k)\}_{k=1,\ldots,n}$ is an *equi-partition* of (x, I) if $(x, I_1) \sim \ldots \sim (x, I_n)$, and $\{I_1, \ldots, I_n\}$ is a partition of I.

Lemma 12 says that we may cut any pair of experiences into n equal pieces by preference, and the ranking between any two pieces, one from each experience, will be the same as between the original pair of experiences.

Lemma 12. *Suppose Axioms 2.1 and 2.2. Let $\{(x, I_k)\}_{k \in K}$ and $\{(y, J_k)\}_{k \in K}$ be equi-partitions of (x, I) and (y, J), respectively, where $K = \{1, \ldots, n\}$ and $n \geq 2$. Then $(x, I) = \sum_{k \in K} (x, I_k) \succeq \sum_{k \in K} (y, J_k) = (y, J)$ if and only if $\forall k, l \in K$, $(x, I_k) \succeq (y, J_l)$.*

Proof. To show $\Rightarrow$, suppose by contradiction $(x, I) \succeq (y, J)$ and $(x, I_a) \prec (y, J_b)$ for some $a, b \in K$. Then $(x, I_k) \prec (y, J_l)$ for all $k, l \in K$, and by Axiom 2.2 $(x, I) = \sum_{k \in K} (x, I_k) \prec \sum_{k \in K} (y, J_k) = (y, J)$, a contradiction. $\Leftarrow$ follows immediately by Axiom 2.2. ■

Recall that $X^+ = \{x \in X : \exists I \in \mathcal{I}, (x,I) \succ (x,\emptyset)\}$, $X^- = \{x \in X : \exists I \in \mathcal{I}, (x,I) \prec (x,\emptyset)\}$, and $\succeq$ is diametric if and only if X^+ and X^- are non-empty and disjoint. We are now ready to prove Theorem 2.

Theorem 2. *Axioms 2.1, 2.2, 2.3, 2.4, and 2.5 hold if and only if there exists a set $V = \{v_x\}_{x \in X}$ of instantaneous utility functions $v_x : [0,1] \to \mathbb{R}$, each of which is continuous and crosses zero finitely many times, such that $\succeq$ is represented by the experienced utility function $U : X \times \mathcal{I} \to \mathbb{R}$ given by*

$$U(x,I) = \int_I v_x(t)\ \mathrm{d}t.$$

If preferences are diametric, $V^+ = \{v_x\}_{x \in X^+}$ and $V^- = \{v_x\}_{x \in X^-}$ are independently ratio-scale.[36] *Otherwise, V is ratio-scale.*

Proof. Let $\succeq_x$ be the relation $\succeq$ restricted to $\{x\} \times \mathcal{I}$. Axioms 2.1, 2.2, 2.3, and 2.4 imply Axioms 1.1, 1.2, 1.3, and 1.4 on $\succeq_x$ for each $x \in X$. By Theorem 1, for each x there exists an experienced utility representation $U_x(I) = \int_I v_x(t)\ \mathrm{d}t$ of $\succeq_x$, unique up to a positive scalar $\alpha_x > 0$. Since preferences are continuous on a compact set, there exists a preference maximal and minimal interval for each $x \in X$. Let $M_x \in \{I \in \mathcal{I} : \forall J \in \mathcal{I},\ (x,I) \succeq (x,J)\}$ denote a maximal interval for x and $m_x \in \{I \in \mathcal{I} : \forall J \in \mathcal{I},\ (x,I) \preceq (x,J)\}$ denote a minimal interval for x. By Axiom 2.5, $\succeq$ agree on empty experiences. Hence, let 0 denote any such empty experience. Let $(X \times \mathcal{I})^+ = \{(x,I) : (x,I) \succ 0\}$ denote the set of positive experiences and $(X \times \mathcal{I})^- = \{(x,I) : (x,I) \prec 0\}$ denote the set of negative experiences.

If X^+ is non-empty, for each $x \in X^+$, let U_x^+ be an experienced utility representations of $\succeq_x$ and normalize $U_x^+(M_x) = 1$. If X^+ is a singleton, $U(x,I) = U_x^+(I)$ represents preferences over $(X \times \mathcal{I})^+$. Otherwise, consider any two experiences $y, z \in X^+$ such that $(y, M_y) \succeq (z, M_z) \succ 0$. Then there exists an interval $L \subseteq M_y$ such that $(y,L) \sim (z, M_z)$. Let $\alpha_{yz}^+ = U_y(L)$ (hence $\alpha_{yz}^+ U_y^+(M_y) = U_y(L) = U_z^+(M_z)$) and $\alpha_{zy}^+ = 1/\alpha_{yz}^+$. This is the ratio of U_y^+ to U_z^+ that "calibrates" the utilities. In particular, we would like to show that

$$\hat{U}^+(x,I) = \begin{cases} \alpha_{yz}^+ U_y^+(I) & \text{if } x = y \\ U_y^+(I) & \text{if } x = z \end{cases}$$

represents preferences over $(\{y,z\} \times \mathcal{I})^+$. For any $(y,J) \succ (y,L) \sim (z,M_z)$, $\hat{U}^+(y,J) > \hat{U}^+(z,K)$ for *all* (z,K). For any $0 \prec (y,J) \preceq (y,L)$, let $(y,J_n) = (y, L_1^n \oplus \ldots \oplus L_{m(n)}^n) \preceq (y,J)$ be the concatenation of the first $m(n)$ elements of some equi-partition of (y,L) with cardinality n and let $\hat{J} = \lim_{n\to\infty} J_n$. Then $(y,\hat{J}) \sim (y,J)$. Let $(z,K_n) = (z, M_{z,1}^n \oplus \ldots \oplus M_{z,m(n)}^n)$ be the concatenation of

[36] $\tilde{U}(x,I) = \int_I \tilde{v}_x(t)\ \mathrm{d}t$ is another representation if and only if for some $\alpha_1, \alpha_2 > 0$, $\tilde{v}_x = \alpha_1 v_x$ for all $x \in X^+$ and $\tilde{v}_x = \alpha_2 v_x$ for all $x \in X^-$.

the first $m(n)$ elements of some equi-partition of (z, M_z) with cardinality n and let $\hat{K} = \lim_{n\to\infty} K_n$. By Lemma 12, $(z, \hat{K}) \sim (y, \hat{J}) \sim (y, J)$ and $U_z^+(\hat{K}) = \alpha_{yz}^+ U_y^+(\hat{J}) = \alpha_{yz}^+ U_y^+(J)$. We may do the same in the other direction for any $(z, K) \succ 0$. Hence, $\hat{U}^+(x, I)$ represents positive experiences for alternatives y, z. Now, for some $o \in X^+$ let

$$U^+(x, I) = \alpha_{xo}^+ U_x^+(I)$$

where $\alpha_{oo}^+ = 1$. We would like to show that $U^+(x, I)$ represents preferences over $(X \times \mathcal{I})^+$. For any $y, z \in X^+$ there exists $I, J, K \in \mathcal{I}$ such that $(o, I) \sim (y, J) \sim (z, K)$. Hence, $U_o^+(I) = \alpha_{yo}^+ U_y^+(J)$, $U_o^+(I) = \alpha_{zo}^+ U_z^+(K)$, $U_y^+(J) = \alpha_{zy}^+ U_z^+(K)$, and $\alpha_{yz}^+ U_y^+(J) = U_z^+(K)$, so $\alpha_{yo}^+/\alpha_{zo}^+ = \alpha_{yz}^+ = 1/\alpha_{zy}^+$ as desired.

If X^- is non-empty, for each $x \in X^-$, let U_x^- be an experienced utility representations of $\succeq_x$ and normalize $U_x^-(m_x) = -1$. Proceed as before. Then $U^-(x, I) = \alpha_{xo}^- U_x^-(I)$ represents preferences over $(X \times \mathcal{I})^-$.

For any $(x, I) \in (X \times \mathcal{I})^+$ and $(y, J) \in (X \times \mathcal{I})^-$, $(x, I) \succ 0 \succ (y, J)$ and $U_x^+(I) > 0 > U_y^-(J)$. Moreover, zero segments are indifferent across experiences by Axiom 2.5. Hence, if X^+ and X^- are non-empty and disjoint (preferences are diametric), then for any $\alpha, \beta > 0$,

$$U(x, I) = \begin{cases} \alpha U^+(x, I) & \text{if } (x, I) \in (X \times \mathcal{I})^+ \\ \beta U^-(x, I) & \text{if } (x, I) \in (X \times \mathcal{I})^- \\ 0 & \text{otherwise} \end{cases}$$

is an experienced utility representation of $\succeq$ over $X \times \mathcal{I}$ and is unique up to two positive scalars. If, on the other hand, X^+ or X^- is empty, then U is defined only for weakly positive or negative segments and so is unique up to a positive scalar. Suppose X^+ and X^- are non-empty and non-disjoint. Then $U(x, I)$ is an experienced utility representation only if $\alpha U^+(x, I) = \beta U^-(x, I)$ for all $x \in X^+ \cap X^-$ and $I \in \mathcal{I}$. Let $\gamma = U^+(x, I)/U^-(x, I)$ for all $x \in X^+ \cap X^-$ and $I \in \mathcal{I}$. Then for any $\alpha > 0$,

$$U(x, I) = \begin{cases} \alpha U^+(x, I) & \text{if } (x, I) \in (X \times \mathcal{I})^+ \\ \alpha\gamma U^-(x, I) & \text{if } (x, I) \in (X \times \mathcal{I})^- \\ 0 & \text{otherwise} \end{cases}$$

is an experienced utility representation of $\succeq$ and is unique up to a positive scalar.

■

C.3 Proof of Theorem 3

By Axiom 3.5, all empty experience singletons are indifferent, so when there is no confusion we may denote any such multiset by 0. Once again, I will sometimes write (x, I) to mean the singleton $\{(x, I)\}$.

Lemma 13 says that we may cut any pair of experiences into n equal pieces by preference, and the ranking between the collection of any two pieces, one from each experience, and the empty experience will be the same as between the collection of the original pair of experiences and the empty experience.

Lemma 13. *Suppose Axioms 3.1, 3.2, and 3.6. Let* $\{(x, I_k)\}_{k\in K}$ *and* $\{(y, J_k)\}_{k\in K}$ *be equi-partitions of* (x, I) *and* (y, J)*, respectively, where* $K = \{1, \ldots, n\}$ *and* $n \geq 2$*. Then* $\{(x, I), (y, J)\} \succeq 0$ *if and only if* $\forall k, l \in K$*,* $\{(x, I_k), (y, J_l)\} \succeq 0$*.*

Proof. To show $\Rightarrow$, suppose by contradiction $\{(x, I), (y, J)\} \succeq 0$ and $\{(x, I_a), (y, J_b)\} \prec 0$ for some $a, b \in K$. By Axiom 3.2, $\{(x, I_k), (y, J_l)\} \prec 0$ for all $k, l \in K$ and $\{(x, I), (y, J)\} \prec 0$, a contradiction. $\Leftarrow$ follows immediately by Axiom 3.2. ■

Lemma 14 says that there exists an experienced utility representation U of preferences over singletons which also represents comparisons of any doubleton to an empty experience.

Lemma 14. *Suppose Axioms 3.1, 3.2, 3.3, 3.4, 3.5, and 3.6. Then there exists an experienced utility representation* $U(x, I)$ *unique up to a positive scalar that represents preferences over singletons and such that for any* $y, z \in X$*,* $\{(y, J), (z, K)\} \succeq 0 \iff U(y, J) + U(z, K) \geq 0$*.*

Proof. Axioms 3.1, 3.2, 3.3, 3.4, and 3.5 imply Axioms 2.1, 2.2, 2.3, 2.4, and 2.5 on $\succeq$ over singletons. For any (x, I), let (x, I_k^n) be the kth element of an equi-partition of (x, I) with cardinality n. We would like to show that for any $y, z \in X$, $\{(y, J), (z, K)\} \succeq 0 \iff U(y, J) + U(z, K) \geq 0$.

Suppose X^+ and X^- are non-empty and disjoint (preference are diametric). By Theorem 2, there exists an experienced utility representation

$$\hat{U}(x, I) = \begin{cases} \alpha U^+(x, I) & \text{if } (x, I) \in (X \times \mathcal{I})^+ \\ \beta U^-(x, I) & \text{if } (x, I) \in (X \times \mathcal{I})^- \\ 0 & \text{otherwise} \end{cases}$$

representing preferences over singletons, unique up to two positive scalars $\alpha, \beta > 0$. Let $o^+ \in X^+$ and $o^- \in X^-$. By Axioms 3.2 and 3.5, $\{0, 0\} \sim \{0\}$. By Axiom 3.3, there exist L^*, M^* such that $\{(o^+, L^*), (o^-, M^*)\} \sim 0$ and for all $L, M \in \mathcal{I}$ such that $\{(o^+, L), (o^-, M)\} \sim 0$, $(o^+, L^*) \succeq (o^+, L)$ and $(o^-, M^*) \preceq (o^-, M)$. Let γ solve $U^+(o^+, L^*) + \gamma U^-(o^-, M^*) = 0$ and for any $\alpha > 0$,

$$U(x, I) = \begin{cases} \alpha U^+(x, I) & \text{if } (x, I) \in (X \times \mathcal{I})^+ \\ \alpha\gamma U^-(x, I) & \text{if } (x, I) \in (X \times \mathcal{I})^- \\ 0 & \text{otherwise} \end{cases}.$$

For any $(o^+, L) \succ (o^+, L^*)$ and any (o^-, M), $U(o^+, L)+U(o^-, M) > 0$ and $\{(o^+, L), (o^-, M)\} \succ 0$ by Axiom 3.2. For any $(o^-, M) \prec (o^-, M^*)$ and any (o^+, L), $U(o^+, L)+U(o^-, M) < 0$ and $\{(o^+, L), (o^-, M)\} \prec 0$ by Axiom 3.2.

For any $(o^+, L) \preceq (o^+, L^*)$, let $(o^+, L_n) = (o^+, L_1^{*n} \oplus \ldots \oplus L_{m(n)}^{*n}) \preceq (o^+, L)$ be the concatenation of the first $m(n)$ elements of some equi-partition of (o^+, L^*) with cardinality n and let $\hat{L} = \lim_{n\to\infty} L_n$. Then $(o^+, \hat{L}) \sim (o^+, L)$. For any $(o^-, M) \succeq (o^-, M^*)$, let $(o^-, M_n) = (o^-, M_1^{*n} \oplus \ldots \oplus M_{m(n)}^{*n}) \preceq (o^-, M)$ be the concatenation of the first $m(n)$ elements of some equi-partition of (o^-, M^*) with cardinality n and let $\hat{M} = \lim_{n\to\infty} M_n$. Then $(o^-, \hat{M}) \sim (o^-, M)$. By Axiom 3.2, $\{(o^+, L), (o^-, M)\} \succeq 0 \iff \{(o^+, \hat{L}), (o^-, \hat{M})\} \succeq 0$ and by Lemma 13, $\{(o^+, \hat{L}), (o^-, \hat{M})\} \succeq 0 \iff U(o^+, L) + U(o^-, M) \geq 0$.

Consider any pair of experiences $(y, J), (z, K) \in X \times \mathcal{I}$. If $(y, J), (z, K) \succ (\prec)\, 0$, then trivially $U(y, J) + U(z, K) \geq 0 \iff \{(y, J), (z, K)\} \succeq 0$. If $(y, J) \succ 0 \succ (z, K)$, then there exist $L, M \in \mathcal{I}$ and $n \in \mathbb{Z}_+$ such that $(o^+, L) \succ 0 \succ (o^-, M)$ and $(y, J_k^n) \sim (o^+, L) \succ 0 \succ (o^-, M) \sim (z, K_k^n)$ for all $k = 1, \ldots, n$. By Lemma 13, $\{(y, J), (z, K)\} \succeq 0 \iff \{(y, J_k^n), (z, K_k^n)\} \succeq 0$, by Axiom 3.2, $\{(y, J_k^n), (z, K_k^n)\} \succeq 0 \iff \{(o^+, L), (o^-, M)\} \succeq 0$, and from above, $\{(o^+, L), (o^-, M)\} \succeq 0 \iff \frac{1}{n}(U(y, J) + U(z, K)) = U(o^+, L) + U(o^-, M) \geq 0$.

Suppose X^+ and X^- are non-empty and non-disjoint. By Theorem 2, there exists an experienced utility representation $U(x, I)$ representing preferences over singletons, unique up to a positive scalar. Let $o \in X^+ \cap X^-$. Consider any pair of experiences $(y, J), (z, K) \in X \times \mathcal{I}$. If $(y, J), (z, K) \succ (\prec)\, 0$, then trivially $U(y, J) + U(z, K) \geq 0 \iff \{(y, J), (z, K)\} \succeq 0$. If $(y, J) \succ 0 \succ (z, K)$, then there exist adjacent intervals $L, M \in \mathcal{I}$ and $n \in \mathbb{Z}_+$ such that $(o, L) \succ 0 \succ (o, M)$ and $(y, J_k^n) \sim (o, L) \succ 0 \succ (o, M) \sim (z, K_k^n)$ for all $k = 1, \ldots, n$. By Lemma 13, $\frac{1}{n}(U(y, J) + U(z, K)) \geq 0 \iff (o, L \oplus M) \succeq 0 \iff \{(y, J_k^n), (z, K_k^n)\} \succeq 0 \iff \{(y, J), (z, K)\} \succeq 0$.

If one or both of X^+ and X^- are empty, the result follows trivially by Axioms 3.2 and 3.5. ∎

We are now ready to prove Theorem 3.

Theorem 3. *Axioms 3.1, 3.2, 3.3, 3.4, 3.5, and 3.6 hold if and only if there exists a set $V = \{v_x\}_{x\in X}$ of instantaneous utility functions $v_x : [0, 1] \to \mathbb{R}$, each of which is continuous and crosses zero finitely many times, such that $\succeq$ is represented by the experienced utility function $\mathcal{U} : \mathcal{M}(X \times \mathcal{I}) \to \mathbb{R}$ given by*

$$\mathcal{U}(A) = \sum_{(x,I)\in A} \int_I v_x(t)\ \mathrm{d}t = \sum_{(x,I)\in A} U(x, I).$$

Moreover, V is ratio-scale.

Proof. Consider any two multisets $A, B \in \mathcal{M}(X \times \mathcal{I})$. We would like to show that the ranking between A and B is pinned down by preferences over singletons and comparisons of doubletons to an empty experience. By Lemma 14, these preferences are represented by an experienced utility function U unique up to a positive scalar.

Given any two experiences $(x, I) \succeq(\preceq)\ (y, J) \succ(\prec) 0$, there exists a partition I_1, I_2 of I such that $(x, I_2) \sim (y, J)$. If $(x, I) \in A$ and $(y, J) \in B$, then $A = A - \{(x, I)\} + \{(x, I_1)\} + \{(x, I_2)\} \succeq B - \{(y, J)\} + \{(y, J)\} = B \iff A' \equiv A - \{(x, I)\} + \{(x, I_1)\} \succeq B - \{(y, J)\} \equiv B'$ by Axiom 3.2. Similarly if $(x, I) \in B$ and $(y, J) \in A$. Hence, we have removed two elements from $A + B$ and added one. We may repeat this as long as there are positive experiences in both A and B or negative experiences in both A and B.

Given any two experiences $(x, I) \succ 0 \succ (y, J)$, if $\{(x, I), (y, J)\} \succeq 0$, there exists a partition I_1, I_2 of I such that $\{(x, I_2), (y, J)\} \sim 0$. If $(x, I), (y, J) \in A$, then $A = A - \{(x, I), (y, J)\} + \{(x, I_1)\} + \{(x, I_2), (y, J)\} \succeq B + \{0\} \iff A' \equiv A - \{(x, I), (y, J)\} + \{(x, I_1)\} \succeq B \equiv B'$ by Axiom 3.2. Similarly if $(x, I), (y, J) \in B$. If $\{(x, I), (y, J)\} \prec 0$, there exists a partition J_1, J_2 of J such that $\{(x, I), (y, J_2)\} \sim 0$. If $(x, I), (y, J) \in A$, then $A = A - \{(x, I), (y, J)\} + \{(x, J_1)\} + \{(x, I), (y, J_2\} \succeq B + \{0\} \iff A' \equiv A - \{(x, I), (y, J)\} + \{(x, J_1)\} \succeq B \equiv B'$ by Axiom 3.2. Similarly if $(x, I), (y, J) \in B$. Hence, we have removed two elements from $A + B$ and added one. We may repeat this as long as there are positive and negative experiences in A or in B.

Since A and B are finite, we may repeat these steps until A contains only positive (negative) experiences (or is empty) and B contains only negative (positive) experiences (or is empty). Preferences are then pinned down by Axiom 3.2. Since U agrees with $\succeq$ at each step, $A \succeq B \iff \sum_{(x,I) \in A} U(x, I) \geq \sum_{(x,I) \in B} U(x, I)$.

■

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